Discrete Mathematics  
Comprehensive Examination, Fall 2005

Sign the exam with your student number - not your name   

Answer the following questions to the best of your ability.

1. (20 pts) Construct a truth table for the Boolean expression \((p \land (p \Rightarrow q)) \Rightarrow q\). Discuss the significance of the expression and its truth values.

   Answer:

   \[
   \begin{array}{c|cccc}
   p & q & p \Rightarrow q & p \land p \Rightarrow q & (p \land (p \Rightarrow q)) \Rightarrow q \\
   \hline
   0 & 0 & 1 & 0 & 1 \\
   0 & 1 & 1 & 0 & 1 \\
   1 & 0 & 0 & 0 & 1 \\
   1 & 1 & 1 & 1 & 1 \\
   \end{array}
   \]

   The expression is a tautology: It is called “modus ponens,” which a primary rule of inference used in direct proofs of statements \(q\) from an assumption \(p\).
2. (20 pts) Consider the “divides” relation on the set of natural numbers \( \mathbb{N} \)
\[
 a \mid b \iff \exists c \in \mathbb{N}, b = ac
\]

1. Show that “divides” is a partial order on the set \( \mathbb{N} \)

**Reflexive:** Divides is reflexive: \( a \mid a \) since \( a = 1 \times a \)

**Symmetric:** Divides is antisymmetric: if \( a \mid b \) and \( b \mid a \), then \( b = c \times a \) and \( a = d \times b \) for some natural numbers \( c \) and \( d \). Therefore
\[
 b = c \times d \times b
\]
which implies \( c = d = 1 \) and \( a = b \).

**Transitive:** Divides is transitive: if \( a \mid b \) and \( b \mid c \), then \( b = d \times a \) and \( c = e \times b \).

Therefore,
\[
 c = e \times b = e \times d \times a
\]
and so \( a \mid c \).

2. (20 pts) Draw a graph (a Hasse diagram) that shows how “divides” orders the numbers

\[ 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20 \]

Answer:

\[
\begin{array}{c}
  \text{4} \\
  \text{6} \\
  \text{9} \\
  \text{10} \\
  \text{14} \\
  \text{8} \\
  \text{12} \\
  \text{18} \\
  \text{16} \\
  \text{20} \\
\end{array}
\]
3. (20 pts) Use (strong) mathematical induction to prove that the number of leaves in a binary tree is \((n + 1)/2\) where \(n\) is the number of nodes (assume integer arithmetic when \(n\) is even).

**Basis:** A binary tree with \(n = 1\) node has \((1 + 1)/2 = 1\) leaf.

**Strong Inductive Hypothesis:** Suppose that all binary trees with \(n\) or fewer nodes have \((n + 1)/2\) leaves, where \(n \geq 1\).

**Inductive Step:** Let \(T\) be a binary tree with \(n + 1\) nodes and let \(L\) and \(R\) be its left and right subtree. Both \(L\) and \(R\) have \(n\) or fewer nodes, say they have \(n_l\) and \(n_r\) nodes where \(n_l + n_r = n\). By the inductive hypothesis, \(L\) has \((n_l + 1)/2\) and \(R\) has \((n_r + 1)/2\) leaves. Therefore \(T\) has

\[
\frac{n_l + 1}{2} + \frac{n_r + 1}{2} = \frac{n_l + n_r + 2}{2} = \frac{n + 1}{2}
\]

leaves.
4. (20 pts) Answer the following questions about binary strings.

1. How many $n$-bit binary strings are there?
   Answer: $2^n$

2. How many $n$-bit binary strings are there with exactly 3 “1” bits?
   Answer: $\binom{n}{3}$ for $n \geq 3$; 0 otherwise.

3. How many $n$-bit binary strings are there with at most 3 “1” bits?
   Answer: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$ for $n \geq 3$; 2, 4 for $n = 1, 2$.

4. How many $n$-bit binary strings have no consecutive “1” bits?

   Answer: Consider how such strings can be formed:

   $$
   \begin{array}{c|ccc}
   n & \epsilon & 0 & 1 \\
   ---&---&---&---
   0 & \epsilon & & \\
   1 & 0 & 1 & \\
   2 & 01 & 00 & 10 \\
   3 & 001 & 101 & 000 & 010 & 100
   \end{array}
   $$

   To get the next list of valid strings append 01 to all strings of length $n-2$: 0101, 0001, 1001 or append 0 to all strings of length $n-1$: 0010, 1010, 0000, 0100, 1000.

   Therefore, the number of strings satisfying the condition is the Fibonacci number $F_{n+2}$, where $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, etc.
5. (20 pts) Consider the program

```c
MM(int n) {
    if (n <= 2) { return 1; }
    else { return 2*MM(n/2) + 2; }
}
```

What is a formula for $\text{MM}(n)$?

Answer:

\[
\begin{align*}
\text{MM}(n) &= 2\text{MM}(n/2) + 2 \\
&= 2(2\text{MM}(n/4) + 2) + 2 \\
&= 2^2(2\text{MM}(n/8) + 2) + 2^2 + 2 \\
&\vdots \\
&= 2^k(2\text{MM}(n/2^{k+1}) + 2) + \sum_{i=1}^{k} 2^i \\
&= 2^k(2 + 2) + \sum_{i=1}^{k} 2^i \\
&= 2^{k+1} + \sum_{i=1}^{k+1} 2^i \\
&= 2^{k+1} + 2^{k+2} - 2
\end{align*}
\]

where $n/2^{k+1} = 2$ or $n = 2^{k+2}$, so that

\[
\text{MM}(n) = 1.5n - 2
\]