INFORMED SEARCH ALGORITHMS

CHAPTER 4, SECTIONS 1–2
Outline

◊ Best-first search
◊ A* search
◊ Heuristics
function Tree-Search( problem, fringe) returns a solution, or failure
  fringe ← Insert(Make-Node(Initial-State[problem]), fringe)
loop do
  if fringe is empty then return failure
  node ← Remove-Front(fringe)
  if Goal-Test[problem] applied to State(node) succeeds return node
  fringe ← InsertAll(Expand(node, problem), fringe)
A strategy is defined by picking the order of node expansion
Best-first search

Idea: use an evaluation function for each node
   – estimate of “desirability”

⇒ Expand most desirable unexpanded node

Implementation:
fringe is a queue sorted in decreasing order of desirability

Special cases:
   greedy search
   A* search
Greedy search

Evaluation function $h(n)$ (heuristic)

$= \text{estimate of cost from } n \text{ to the closest goal}$

E.g., $h_{SLD}(n) = \text{straight-line distance from } n \text{ to Bucharest}$

Greedy search expands the node that appears to be closest to goal
Greedy search example
Greedy search example

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Greedy search example

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Greedy search example
Properties of greedy search

Complete??
Properties of greedy search

**Complete??** No—can get stuck in loops, e.g., with Oradea as goal,

Iasi $\rightarrow$ Neamt $\rightarrow$ Iasi $\rightarrow$ Neamt $\rightarrow$

Complete in finite space with repeated-state checking

**Time??**
Properties of greedy search

**Complete**? No—can get stuck in loops, e.g.,

Iasi → Neamt → Iasi → Neamt →

Complete in finite space with repeated-state checking

**Time**? $O(b^m)$, but a good heuristic can give dramatic improvement

**Space**?
Properties of greedy search

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Time? $O(b^m)$, but a good heuristic can give dramatic improvement

Space? $O(b^m)$—keeps all nodes in memory

Optimal??
Properties of greedy search

**Complete**? No—can get stuck in loops, e.g.,

\[ \text{Iasi} \rightarrow \text{Neamt} \rightarrow \text{Iasi} \rightarrow \text{Neamt} \rightarrow \]

Complete in finite space with repeated-state checking

**Time**? \( O(b^m) \), but a good heuristic can give dramatic improvement

**Space**? \( O(b^m) \)—keeps all nodes in memory

**Optimal**? No
A* search

Idea: avoid expanding paths that are already expensive

Evaluation function \( f(n) = g(n) + h(n) \)

\( g(n) \) = cost so far to reach \( n \)
\( h(n) \) = estimated cost to goal from \( n \)
\( f(n) \) = estimated total cost of path through \( n \) to goal

A* search uses an admissible heuristic
i.e., \( h(n) \leq h^*(n) \) where \( h^*(n) \) is the true cost from \( n \).
(Also require \( h(n) \geq 0 \), so \( h(G) = 0 \) for any goal \( G \).)

E.g., \( h_{SLD}(n) \) never overestimates the actual road distance

Theorem: A* search is optimal
A* search example

\[ 366 = 0 + 366 \]
A* search example

A* search example

Chapter 4, Sections 1-2
A* search example

Chapter 4, Sections 1-2
A* search example

- Arad
  - Fagaras
  - Oradea
  - Rimnicu Vilcea
    - Craiova
    - Pitesti
    - Sibiu
- Sibiu
- Timisoara
  - 447 = 118 + 329
- Zerind
  - 449 = 75 + 374

Arad
- Fagaras
- Oradea
- Rimnicu Vilcea
  - Craiova
  - Pitesti
  - Sibiu

Arad

646 = 280 + 366
415 = 239 + 176
671 = 291 + 380
526 = 366 + 160
417 = 317 + 100
553 = 300 + 253
**A* search example**

- Arad
- Sibiu
- Timisoara
- Zerind

**Cities and Distances:**
- Arad: 646 = 280 + 366
- Fagaras: 591 = 338 + 253
- Oradea: 671 = 291 + 380
- Rimnicu Vilcea: 646 = 280 + 366
- Bucharest: 553 = 300 + 253
- Craiova: 526 = 366 + 160
- Pitesti: 417 = 317 + 100
- Sibiu: 591 = 338 + 253
- Timisoara: 447 = 118 + 329
- Zerind: 449 = 75 + 374
A* search example

Chapter 4, Sections 1–2
Optimality of A* (standard proof)

Suppose some suboptimal goal $G_2$ has been generated and is in the queue. Let $n$ be an unexpanded node on a shortest path to an optimal goal $G_1$.

\[ f(G_2) = g(G_2) \] since \( h(G_2) = 0 \)
\[ > g(G_1) \] since $G_2$ is suboptimal
\[ \geq f(n) \] since $h$ is admissible

Since $f(G_2) > f(n)$, A* will never select $G_2$ for expansion.
Optimality of A* (more useful)

Lemma: A* expands nodes in order of increasing $f$ value

Gradually adds "$f$-contours" of nodes (cf. breadth-first adds layers)
Contour $i$ has all nodes with $f = f_i$, where $f_i < f_{i+1}$
Properties of A* Complete??
## Properties of A*:

**Complete??** Yes, unless there are infinitely many nodes with $f \leq f(G')$.

**Time??**
Properties of A*

**Complete**  Yes, unless there are infinitely many nodes with $f \leq f(G')$

**Time**  Exponential in [relative error in $h \times$ length of soln.]

**Space**
<table>
<thead>
<tr>
<th>Properties of A*</th>
</tr>
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<tbody>
<tr>
<td><strong>Complete</strong>?</td>
</tr>
<tr>
<td><strong>Time</strong>?</td>
</tr>
<tr>
<td><strong>Space</strong>?</td>
</tr>
</tbody>
</table>
| **Optimal**? | }
## Properties of A* 

**Complete** Yes, unless there are infinitely many nodes with \( f \leq f(G) \)

**Time** Exponential in \([\text{relative error in } h \times \text{length of soln.}]\)

**Space** Keeps all nodes in memory

**Optimal** Yes—cannot expand \( f_{i+1} \) until \( f_i \) is finished

\( A^* \) expands all nodes with \( f(n) < C^* \)
\( A^* \) expands some nodes with \( f(n) = C^* \)
\( A^* \) expands no nodes with \( f(n) > C^* \)
Proof of lemma: Consistency

A heuristic is **consistent** if

\[ h(n) \leq c(n, a, n') + h(n') \]

If \( h \) is consistent, we have

\[
\begin{align*}
f(n') &= g(n') + h(n') \\
&= g(n) + c(n, a, n') + h(n') \\
&\geq g(n) + h(n) \\
&= f(n)
\end{align*}
\]

I.e., \( f(n) \) is nondecreasing along any path.
Admissible heuristics

E.g., for the 8-puzzle:

\[ h_1(n) = \text{number of misplaced tiles} \]
\[ h_2(n) = \text{total Manhattan distance} \]

(i.e., no. of squares from desired location of each tile)

\[
\begin{array}{ccc}
7 & 2 & 4 \\
5 & 6 & \\
8 & 3 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & \\
\end{array}
\]

Start State

Goal State

\[
\begin{align*}
h_1(S) & = ?? \\
h_2(S) & = ??
\end{align*}
\]
Admissible heuristics

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\quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & \\
\end{array}
\]

\[
h_1(S) = 6
\]
\[
h_2(S) = 4 + 0 + 3 + 3 + 1 + 0 + 2 + 1 = 14
\]
Dominance

If \( h_2(n) \geq h_1(n) \) for all \( n \) (both admissible)
then \( h_2 \) dominates \( h_1 \) and is better for search

Typical search costs:

\[
\begin{align*}
  d = 14 & \quad \text{IDS} = 3,473,941 \text{ nodes} \\
  & \quad A^*(h_1) = 539 \text{ nodes} \\
  & \quad A^*(h_2) = 113 \text{ nodes}
\end{align*}
\]

\[
\begin{align*}
  d = 24 & \quad \text{IDS} \approx 54,000,000,000 \text{ nodes} \\
  & \quad A^*(h_1) = 39,135 \text{ nodes} \\
  & \quad A^*(h_2) = 1,641 \text{ nodes}
\end{align*}
\]

Given any admissible heuristics \( h_a, h_b \),

\[
h(n) = \max(h_a(n), h_b(n))
\]

is also admissible and dominates \( h_a, h_b \)
Relaxed problems

Admissible heuristics can be derived from the exact solution cost of a relaxed version of the problem.

If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution.

If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution.

Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem.
Well-known example: travelling salesperson problem (TSP)
Find the shortest tour visiting all cities exactly once

Minimum spanning tree can be computed in $O(n^2)$
and is a lower bound on the shortest (open) tour
Summary

Heuristic functions estimate costs of shortest paths

Good heuristics can dramatically reduce search cost

Greedy best-first search expands lowest $h$
  - incomplete and not always optimal

$A^*$ search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)

Admissible heuristics can be derived from exact solution of relaxed problems