Relations and Functions

Let A and B be two sets. The *cartesian product* of A and B, denoted $A \times B$, is defined to be the set

$$\{(a, b) \mid a \in A \text{ and } b \in B\}$$

For example, if $A = \{a, b, c\}$ and $B = \{a, c, d\}$, then

$$\{(a, a), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, c), (c, d)\}$$

A *binary relation* on A and B is a subset of $A \times B$. For example, for the above two sets the following is a binary relation:

$$R = \{(a, a), (a, d), (b, c)\}$$

A binary relation R on A and B is said to be a *partial function* from A to B if for every element $a \in A$, there is *at most* one element $b \in B$ such that $(a, b) \in R$. In such a case we use the notation $f(a) = b$. For example, the above relation R is not a partial function from A to B, however, the following relation is:

$$R' = \{(a, a), (b, c)\}$$

If f is a partial function from A to B then the *domain* of f is

$$\{a \mid a \in A \text{ and } \exists b \in B \text{ where } f(a) = b\}$$

In addition, the *range* of f is

$$\{b \mid b \in B \text{ and } \exists a \in A \text{ where } f(a) = b\}$$

A binary relation R on A and B is said to be a *total function* if for every element $a \in A$, there is *exactly* one element $b \in B$ such that $(a, b) \in R$. For example, the above relation R' is not a total function, however, the following relation is:

$$R'' = \{(a, a), (b, c), (c, d)\}$$

For our purposes, a total function will be referred to simply as a *function*.

A function f from A to B is said to be *one to one* if $f(a) \neq f(b)$, for all $a, b \in A$ where $a \neq b$. A function f from A to B is said to be *onto* if for each $b \in B$ there exists an $a \in A$ such that $f(a) = b$. A function f from A to B is said to be a *bijection* if it is *one to one* and *onto*.

If R is a binary relation on sets A and B then the *inverse* of R, denoted by R^{-1}, is the set

$$\{(b, a) \mid (a, b) \in R\}$$

Given the above definitions, the following observations can be made.

- The definition of *domain* given in Sudkamp's book is different from that given above. Specifically, he defines the domain of a partial function to be the entire set A.
- Every total function is a partial function.
- A partial function f from A to B is a total function iff (if and only if) the domain of f is equal to A.
- A function f from A to B is *onto* iff the range of f equals B.
- A *one to one* function is not necessarily *onto*. Similarly, an *onto* function is not necessarily *one to one*.
- The domain of a function from A to B is the entire set A.
- If a binary relation R is a function, then its inverse R^{-1} is *not* necessarily a function.
- If there is a bijection f from A to B then the inverse f^{-1} is a bijection from B to A (prove this as an exercise).

The sets A and B have the same *cardinality* iff there is a bijection (a one to one and onto function) from A to B (or from B to A.)

Let **N**={0,1,2,...} denote the natural numbers. Then a set is *countably infinite* iff it has the same cardinality as **N**. A set is *countable* iff it is finite or countably infinite. A set that is not *countable* is said to be *uncountable*.

- If A and B are finite sets and $A \subset B$ then A and B have different cardinality.
- If A and B are infinite sets and $A \subset B$ then A and B can still have the same cardinality!
- As another definition, a set is *countably infinite* iff it has the same cardinality as the set of integers. Prove that this definition is equivalent to the one given above.
- Given the above definitions, how would one prove that an arbitrary set is or is not countably infinite?