# Gauss Jordan Elimination

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# Overview

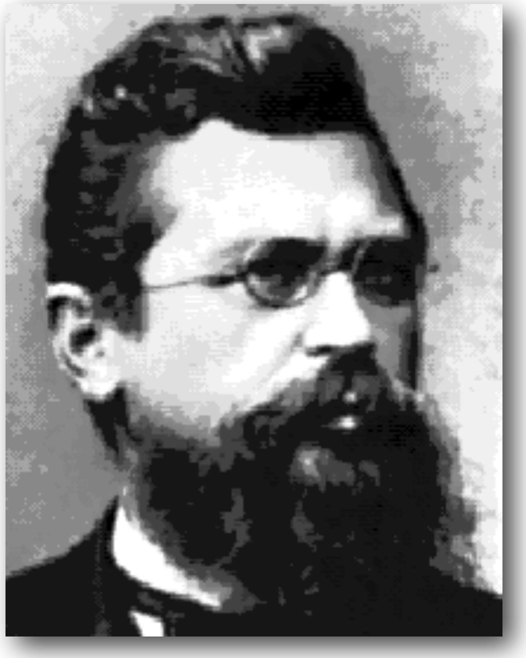
- Background/Description
- Algorithm
- Code snippets
- Examples
- Analysis

# Background/Description

# Background

- Named for Carl Friedrich Gauss and Wilhelm Jordan
- Started out as "Gaussian elimination" although Gauss didn't create it
- Jordan improved it in 1887 because he needed a more stable algorithm for his surveying calculations

Carl Gauss
mathematician/scientist
1777-1855



Wilhelm Jordan
geodesist
1842-1899

(geodesy involves taking measurements of the Earth)

# Some Terms

- Matrix - 2D array
- Identity matrix - Matrix with all 0s except for 1s on the diagonal
- Determinant - Representative number that can be calculated from a matrix
- Matrix inverse - The matrix version of n^-1

| | | | |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 |

A 4x4 identity matrix

# Elementary Operations

- Steps that can be performed on matrices without changing their overall meaning
- Multiplying by a scalar - Replace a row/column by itself times a factor
- Linear combinations - Replace a row/column by a combination of itself and another row/column
- Pivoting - Interchanging two rows/columns
  - Don’t need pivoting but it really helps

# Gaussian Elimination

- First seen used in the Chinese text "The Nine Chapters on the Mathematical Art" and in Isaac Newton's notes
- Puts a matrix into *row echelon form*, and then uses back substitution to solve
- Determinant is product of diagonals

$$\begin{matrix} 2 & 4 & 1 & 4 & 7 \\ 0 & 6 & 3 & 5 & 3 \\ 0 & 0 & 2 & 6 & 9 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{matrix}$$

Row echelon form:
Lower triangle is 0s

# Gauss-Jordan Elimination

- Gauss-Jordan elimination is a faster way to solve matrices and find a matrix inverse
- Puts the matrix into *row-reduced echelon form*

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Reduced row echelon form:
Non-diagonals are 0s

# Comparison

| | Solves system | Finds determinant | Finds inverse | Form used |
|---|---|---|---|---|
| Gauss Elim. | ✓ | ✓ | ✓ | Row Echelon |
| Gauss-Jordan Elim. | ✓ | | ✓ | Reduced Row Echelon |

# Advantages of G-J Elim.

- Can produce both the solution for a set of linear equations and the matrix inverse
- As efficient as most methods when it comes to finding a matrix inverse
- Solving the system of equations doesn't take up that much more time than finding the inverse
- Fairly stable

# Disadvantages of G-J Elim.

- Requires more storage (bookkeeping and right hand elements)
- Takes three times as long than most methods when solving for a single set

# Algorithm

# Algorithm

- Repeat n times, where n is the number of columns
  - Locate a pivot
  - Move the row containing the pivot so that the pivot is on a diagonal
  - Divide the pivot's row by the value of the pivot
  - Subtract multiples of the pivot's row from the rows above and below to make them 0
  - If solving a system of equations, make sure to do the same operations on the vector matrix as well
- Input matrix is replaced by inverse and vector matrix is replaced by solutions

# What is a Pivot?

- A "special" element of a matrix, chosen to become part of the final diagonal
- The pivot is usually the largest element in an unaltered row/column
- Choose a large pivot because that makes it easier to reduce the rest of the row/column

# Code Snippets

# Choosing a pivot

```
for (int i = 0; i < n; i++) {
double big = 0.0;
int icol = 0;
int irow = 0;
// Search for a pivot element in each column
for (int j = 0; j < n; j++) {
   // Check that the column hasn't been visited
   if (ipiv[j] != 1) {
      // Now check through each member of the column
      for (int k = 0; k < n; k++) {
         if (ipiv[k] == 0) {
            if (fabs(a.get(j, k)) >= big) {
               big = fabs(a.get(j, k));
               irow = j;
               icol = k;
            }
         }
      }
   }
}
```

Essentially chooses the largest (absolute value) element on an unvisited column and row

# Moving To Diagonal

```
// Interchange rows to put the pivot on the diagonal
      if (irow != icol) {
         a.exchange_rows(irow, icol);
         b.exchange_rows(irow, icol);
         if (verbose) {
            printf("Exchanging rows %d and %d\n", irow,
icol);
            a.print();
         }
      }
```

Swaps rows so that the pivot's row number and column number are equal

# Normalizing row

Will explain this in a bit

```
// Divide the row by the pivot
      double pivot_inverse = 1.0 / a.get(icol, icol);
      a.set(icol, icol, 1.0);
      a.mult_row(icol, pivot_inverse);
      b.mult_row(icol, pivot_inverse);
      if (verbose) {
         printf("Dividing row %d by %.2f\n", icol, 1.0 /
pivot_inverse);
         a.print();
      }
```

Divides the pivot's row by the value of the pivot.

# Reducing column

Will explain this in a bit →

```
// Reduce the rows (except for the pivot row)
for (int ll = 0; ll < n; ll++) {
   if (ll != icol) {
      double dummy = a.get(ll, icol);
      a.set(ll, icol, 0.0);
      a.add_rows(1.0, ll, -dummy, icol);
      b.add_rows(1.0, ll, -dummy, icol);
      if (verbose) {
         printf("Row %d -= %.2f * Row %d\n", ll, dummy, icol);
         a.print();
      }
   }
}
```

Subtracts multiples of the pivot row from the rows above/below to make the column mostly 0s

# Note: Storage

(i.e. the part that I said I would explain)

- The code in the textbook "saves space" by not storing the identity matrix as a separate matrix. Instead, it coexists with the input matrix.
- This can be done because we know that the input matrix will eventually become the identity matrix.
- That's why the code changes the input matrix to the identity matrix right before doing any replacements

# Simple Example

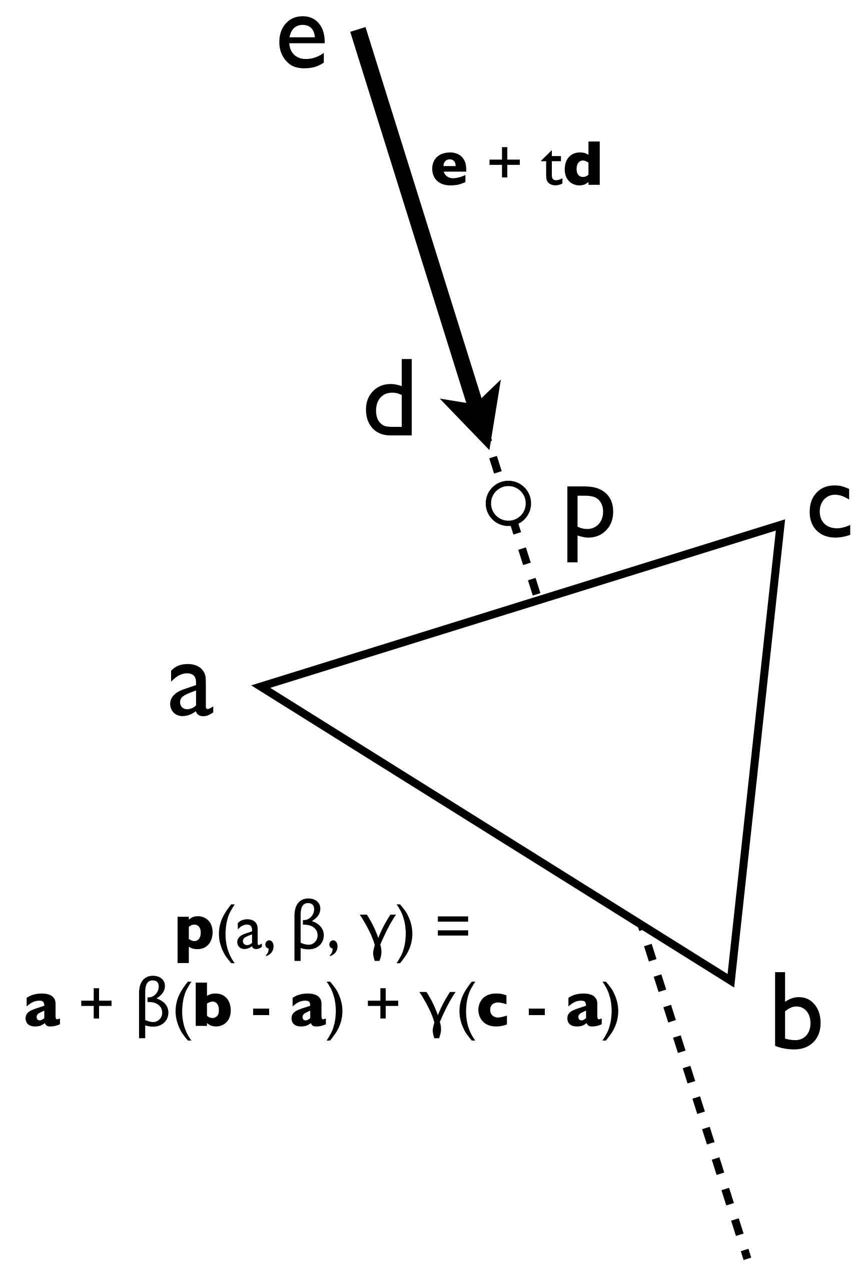
$$x + 2y = 8$$
$$3x + 4y = 20$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \end{bmatrix}$$

| | Matrix | Description |
|---|---|---|
| 1 | 1 2 \| 8<br>3 4 \| 20 | Start |
| 2 | 1 2 \| 8<br>.75 1 \| 5 | Row 1 /= 4 |
| 3 | -.5 0 \| -2<br>.75 1 \| 5 | Row 0 -= 2 * Row 1 |
| 4 | 1 0 \| 4<br>.75 1 \| 5 | Row 0 /= -0.5 |
| 5 | 1 0 \| 4<br>0 1 \| 2 | Row 1 -= 0.75 * Row 0 |

# Real-World Example: Ray-Triangle Intersection

- From Shirley's "Fundamentals of Computer Graphics"
- Goal: Find the point where a ray intersects a plane defined by a triangle
- Basic form of ray tracing




The ray marked by ED intersects the plane defined by triangle ABC at point P

# Ray-Triangle Intersection

Point must lie on both the vector, represented by:

$$\vec{p} = \vec{e} + t\vec{d}$$

and the plane of the triangle, represented by:

$$\vec{p} = \vec{a} + \beta(\vec{b} - \vec{a}) + \gamma(\vec{c} - \vec{a})$$

so the resulting equation to solve is:

$$\vec{e} + t\vec{d} = \vec{a} + \beta(\vec{b} - \vec{a}) + \gamma(\vec{c} - \vec{a})$$

for some t, β, and γ.

# Ray-Triangle Intersection

$$\vec{e} + t\vec{d} = \vec{a} + \beta(\vec{b} - \vec{a}) + \gamma(\vec{c} - \vec{a})$$

In xyz coordinates, this becomes:

$$x_e + tx_d = x_a + \beta(x_b - x_a) + \gamma(x_c - x_a)$$

$$y_e + ty_d = y_a + \beta(y_b - y_a) + \gamma(y_c - y_a)$$

$$z_e + tz_d = z_a + \beta(z_b - z_a) + \gamma(z_c - z_a)$$

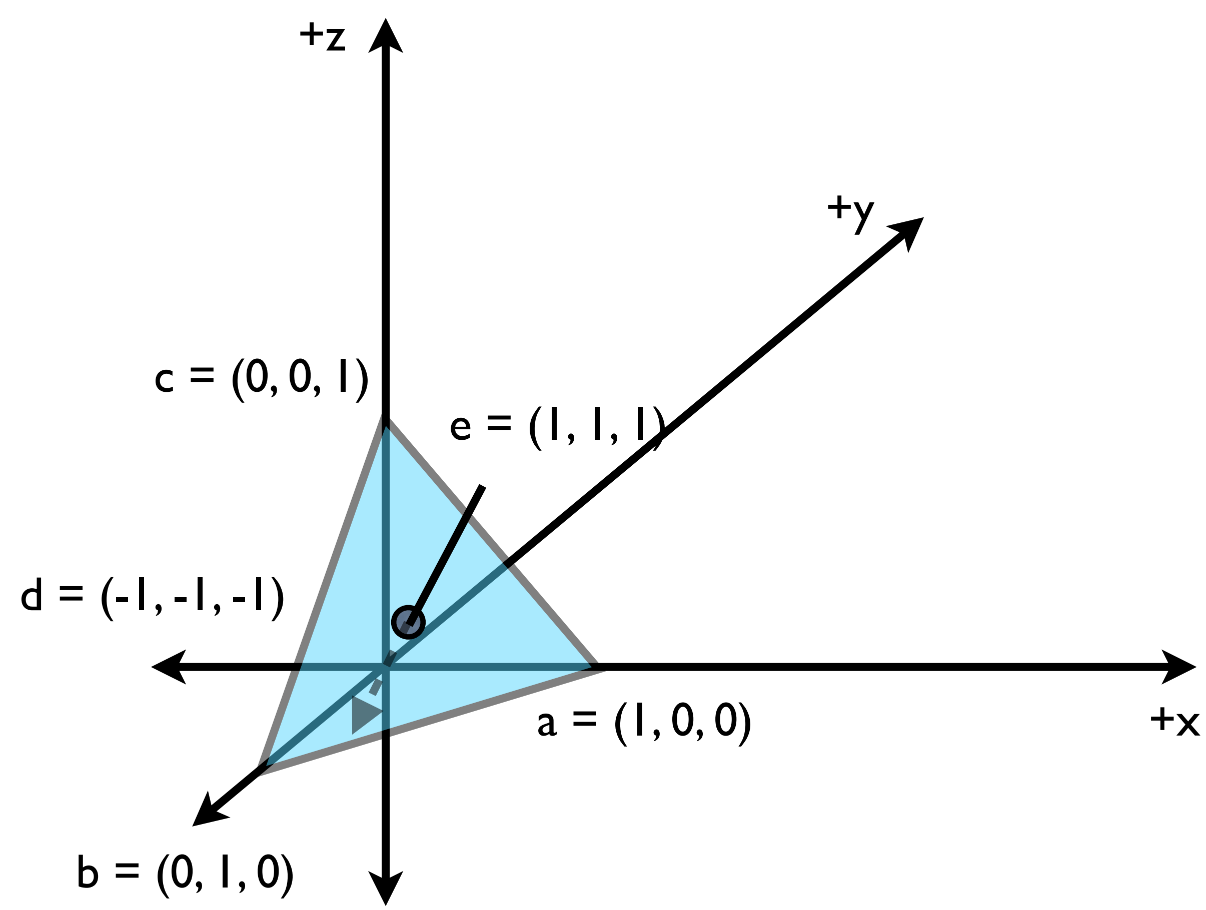
which can also be written as

$$\begin{bmatrix} x_a - x_b & x_a - x_c & x_d \\ y_a - y_b & y_a - y_c & y_d \\ z_a - z_b & z_a - z_c & z_d \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} x_a - x_e \\ y_a - y_e \\ z_a - z_e \end{bmatrix}$$

which can now be solved by Gauss-Jordan elimination!

# Ray-Triangle Intersection

- Find where the ray
(1, 1, 1) + t(-1, -1, -1)
hits a triangle with vertices
(1, 0, 0), (0, 1, 0), and (0, 0, 1)



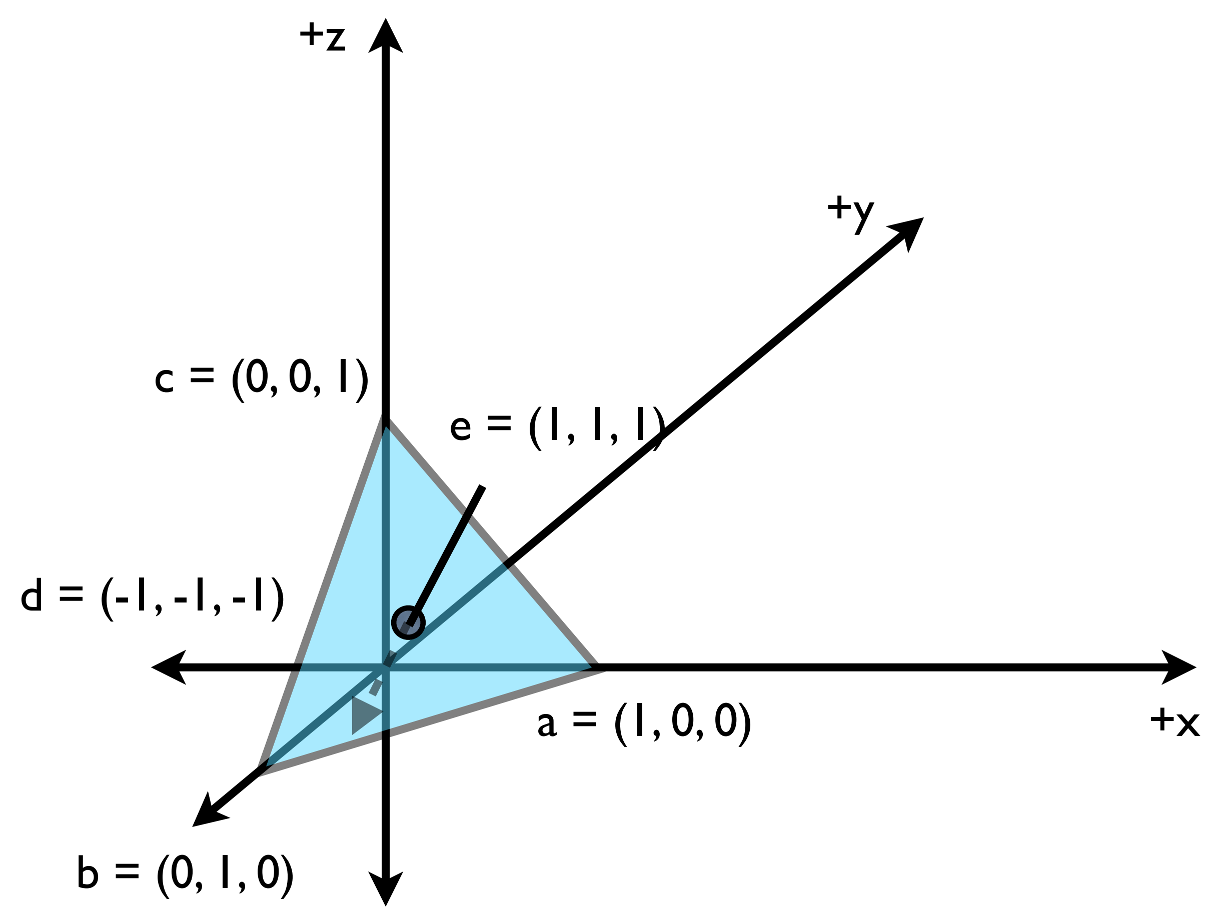
# Ray-Triangle Intersection

Using the previous formula

$$\begin{bmatrix} x_a - x_b & x_a - x_c & x_d \\ y_a - y_b & y_a - y_c & y_d \\ z_a - z_b & z_a - z_c & z_d \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} x_a - x_e \\ y_a - y_e \\ z_a - z_e \end{bmatrix}$$

$$\begin{bmatrix} 1-0 & 1-0 & -1 \\ 0-1 & 0-0 & -1 \\ 0-0 & 0-1 & -1 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} 1-1 \\ 0-1 \\ 0-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$




End result is
t = 2/3
β = 1/3
γ = 1/3

or p = (1/3, 1/3, 1/3)

# Ray-Triangle Intersection

```
input matrix:
1.00    1.00    -1.00
-1.00   0.00    -1.00
0.00    -1.00   -1.00
vector matrix:
0.00
-1.00
-1.00
Starting Gauss-Jordan algorithm...
Dividing row 2 by -1.00
1.00    1.00    -1.00
-1.00   0.00    -1.00
-0.00   1.00    -1.00
Row 0 -= -1.00 * Row 2
1.00    2.00    -1.00
-1.00   0.00    -1.00
-0.00   1.00    -1.00
Row 1 -= -1.00 * Row 2
1.00    2.00    -1.00
-1.00   1.00    -1.00
-0.00   1.00    -1.00
Exchanging rows 0 and 1
-1.00   1.00    -1.00
1.00    2.00    -1.00
-0.00   1.00    -1.00
```

```
Dividing row 1 by 2.00
-1.00   1.00    -1.00
0.50    0.50    -0.50
-0.00   1.00    -1.00
Row 0 -= 1.00 * Row 1
-1.50   -0.50   -0.50
0.50    0.50    -0.50
-0.00   1.00    -1.00
Row 2 -= 1.00 * Row 1
-1.50   -0.50   -0.50
0.50    0.50    -0.50
-0.50   -0.50   -0.50
Dividing row 0 by -1.50
-0.67   0.33    0.33
0.50    0.50    -0.50
-0.50   -0.50   -0.50
Row 1 -= 0.50 * Row 0
-0.67   0.33    0.33
0.33    0.33    -0.67
-0.50   -0.50   -0.50
Row 2 -= -0.50 * Row 0
-0.67   0.33    0.33
0.33    0.33    -0.67
-0.33   -0.33   -0.33
```

```
Exchanging columns 0 and 1
0.33    -0.67   0.33
0.33    0.33    -0.67
-0.33   -0.33   -0.33
Done!
inverse:
0.33    -0.67   0.33
0.33    0.33    -0.67
-0.33   -0.33   -0.33
solution:
0.33
0.33
0.67
```

Code output

# Ray-Triangle Intersection

Substitute 2/3 for t to find p

$$\vec{p} = \vec{e} + t\vec{d}$$

$$\vec{p} = (1,1,1) + \frac{2}{3}(-1,-1,-1) = \boxed{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)}$$

Get the same result with β = γ = 1/3

$$\vec{p} = \vec{a} + \beta(\vec{b} - \vec{a}) + \gamma(\vec{c} - \vec{a})$$

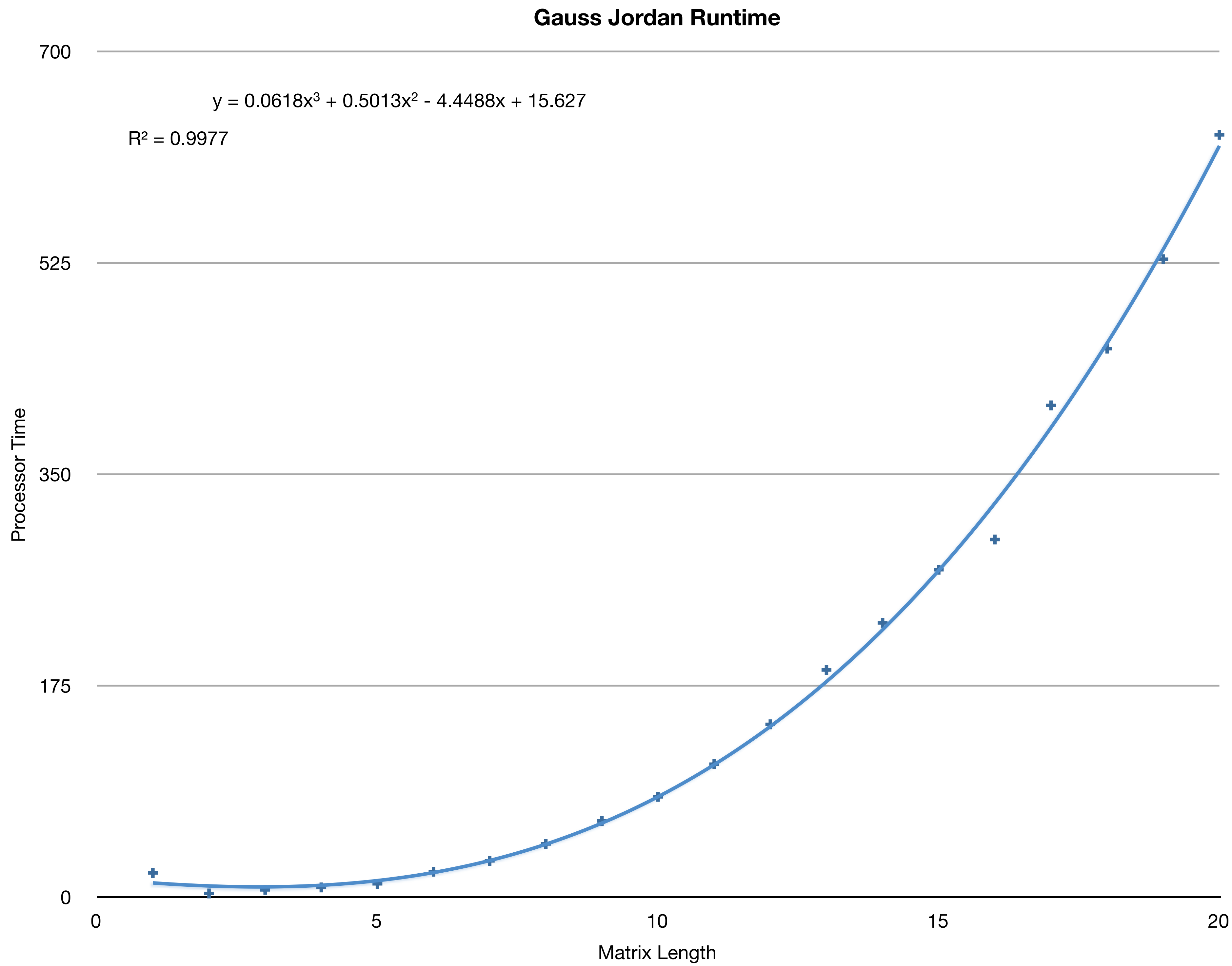
$$\vec{p} = (1,0,0) + \frac{1}{3}(-1,1,0) + \frac{1}{3}(-1,0,1) = \boxed{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)}$$

# Analysis

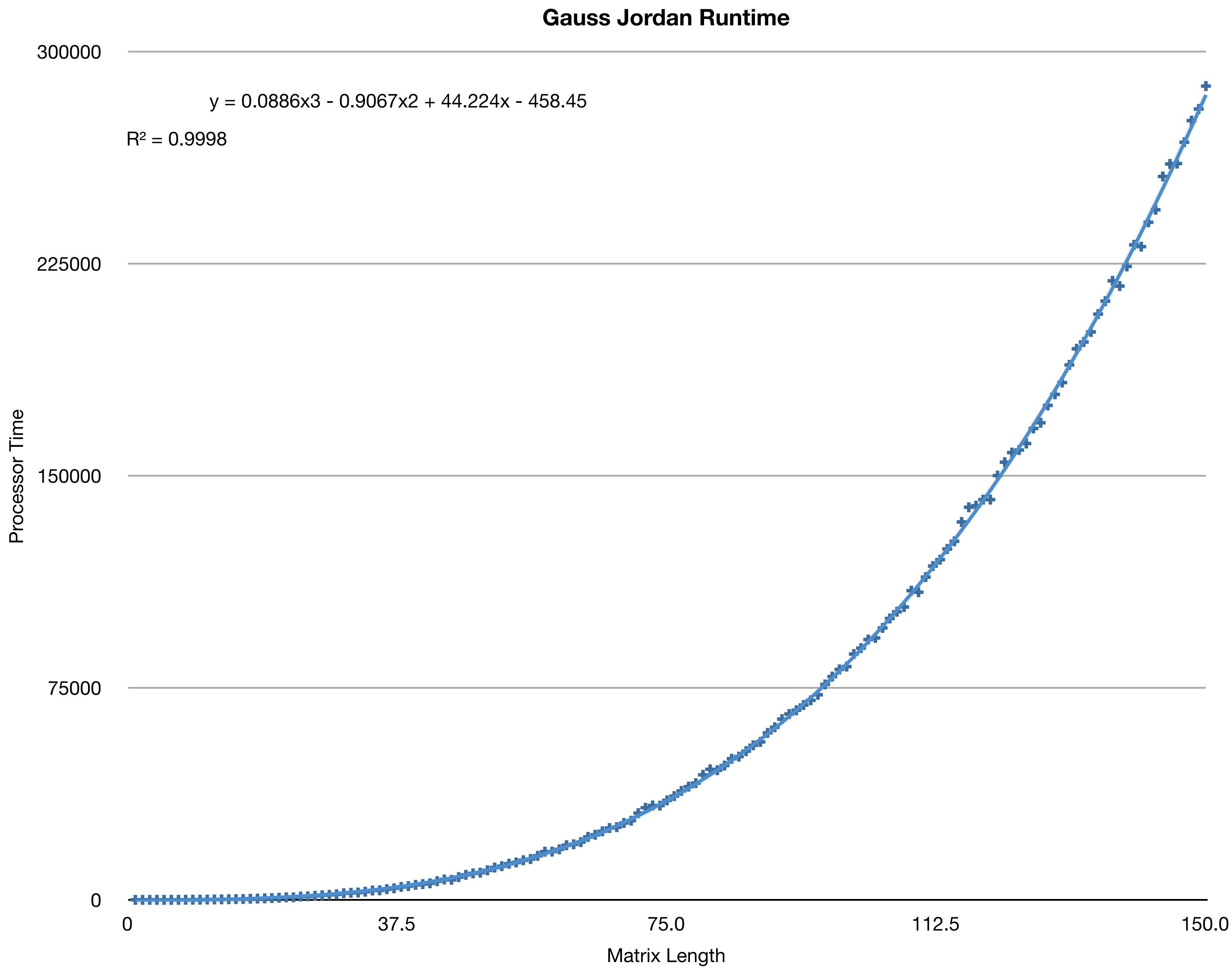
# Analysis

- Used the clock() function to time how long it took to calculate the inverse of matrices of size n
- Matrices were randomly generated
- Took an unusual amount of time to calculate for n=1
- Follows an O(n^3) pattern

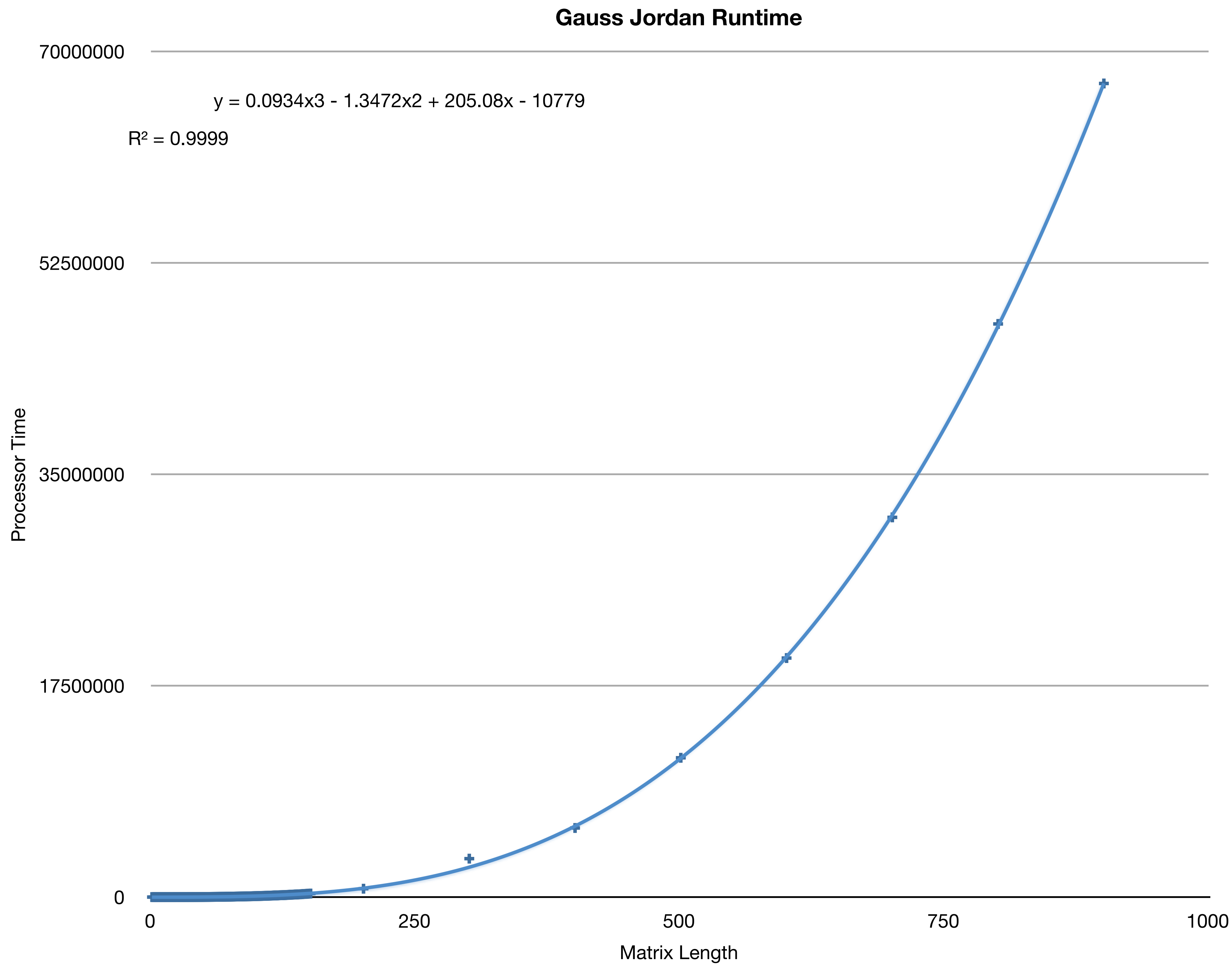
# Efficiency (1,20)



# Efficiency (1, 150)



# Efficiency (1, 900)



# Areas for future analysis

- Investigate the impact of pivoting on processing time
- Compare against other methods for calculating inverses/solving systems of equations

# Sources

- Wikipedia
- Numerical Recipes (Press)
- Fundamentals of Computer Graphics (Shirley)