Rational Function Interpolation
Barycentric Rational Interpolation
Coefficients of Interpolating Polynomial

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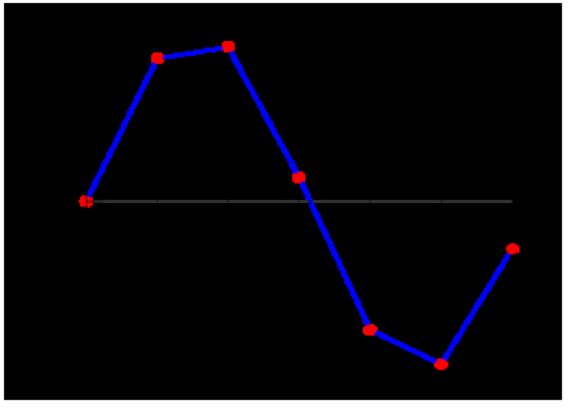
Summary of Previous Interpolation Methods

Linear Interpolation
Polynomial Interpolation
Cubic Spline Interpolation

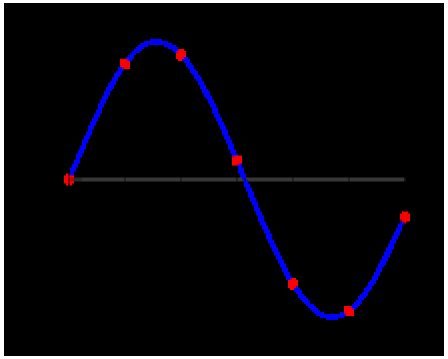
Linear Interpolation

- Concatenation of linear interpolants between a pair of data points
- Piecewise linear function
- Cheap . . . O(NLog(N))
- Connect data points in a table (e.g. given population in 1990 and 2000, what was population in 1995?)
- Historically used with astronomical data



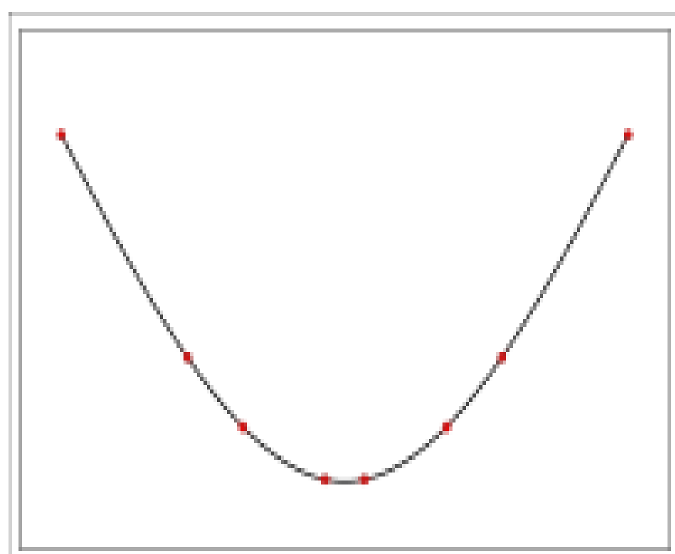
Polynomial Interpolation

- Given a data set, find a polynomial that goes exactly through each point
- Neville's algorithm: $O(N^2)$
- Basis for algorithms solving numerical ODEs and numerical integration
- Unstable on equidistant grid



Cubic Spline

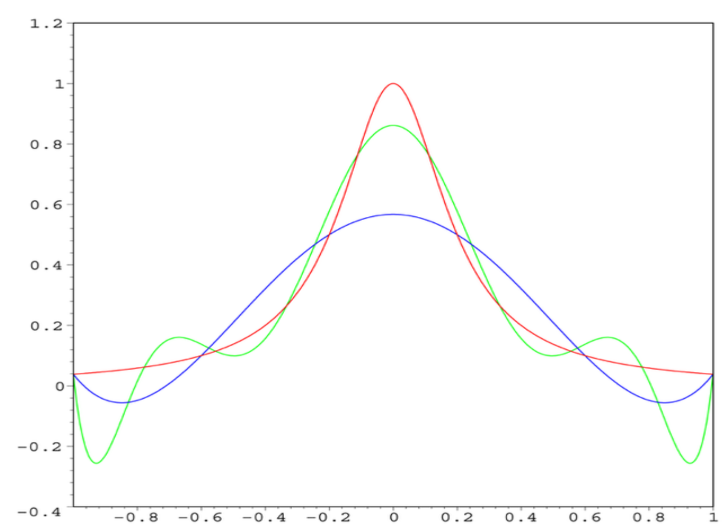
- Interpolation interval divided into subintervals and each subinterval interpolated using 3rd degree polynomial
- Piecewise cubic functions with continuous first and second derivatives
- Requires function continuity and passing through all data points
- O(N) complexity . . . stable and simplistic calculation



Cubic Spline Continued

- Known boundary first derivatives . . . $O(h^4)$
- Natural spline (2nd derivatives = 0) . . . $O(h^2)$
- Preferable to polynomials because the interpolation error can be made small even when using low degree polynomials
- Avoid Runge's Phenomenon in which boundaries oscillate wildly for functions like

$$f(x) = \frac{1}{1 + 25x^2}.$$



Why Rational Function Interpolation?

- Express more diverse behavior than polynomials
- Solves disadvantages of polynomial interpolation but polynomial can be found at any point; rational function cannot
- Ability to model equations with poles (while polynomials fail) , . . Good if modeling a function with poles. Bad if goal is numerical stability.
- Higher orders give higher accuracy

Example of Unstable Rational Interpolation

Numerator and Denominator of 2nd degree

Pole at x = -0.5

x y

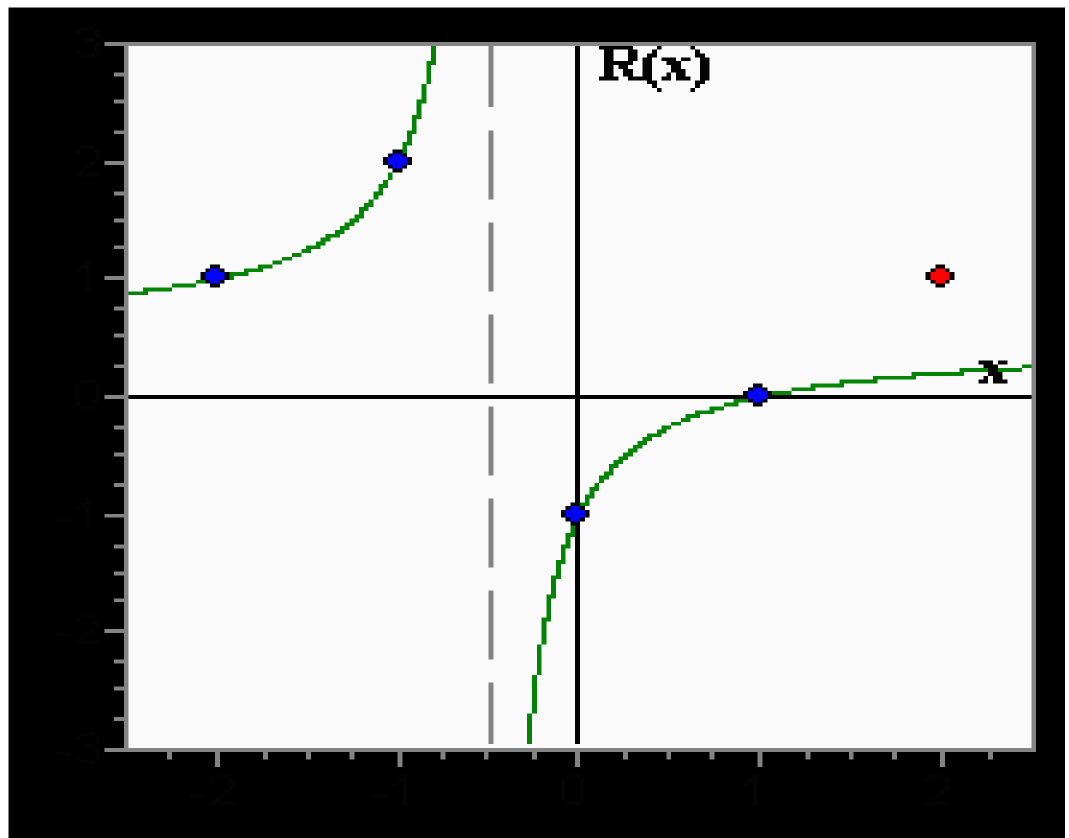
-2 1

-1 2

0 -1

1 0

2 1



Rational Functions Continued

- Historically, the rational interpolant was constructed by solving a set of equations with unknown coefficients.
  - However, the larger the data set the larger the error in calculating coefficients
- Neville's algorithm solves this problem by setting degree of numerator and denominator equal to N/2
- Main disadvantage: no mechanism to find poles

Why Barycentric Rational Interpolation?

- Suppresses all nearby poles
- Experimentation with higher orders encouraged
- Favorable comparison to splines but with smaller error and infinitely smooth curves
- If spacing of points is O(h), error is $O(h^{d+1})$ as $h \rightarrow 0$. The complexity is order O(Nd)

Runge's example with barycentric rational interpolation

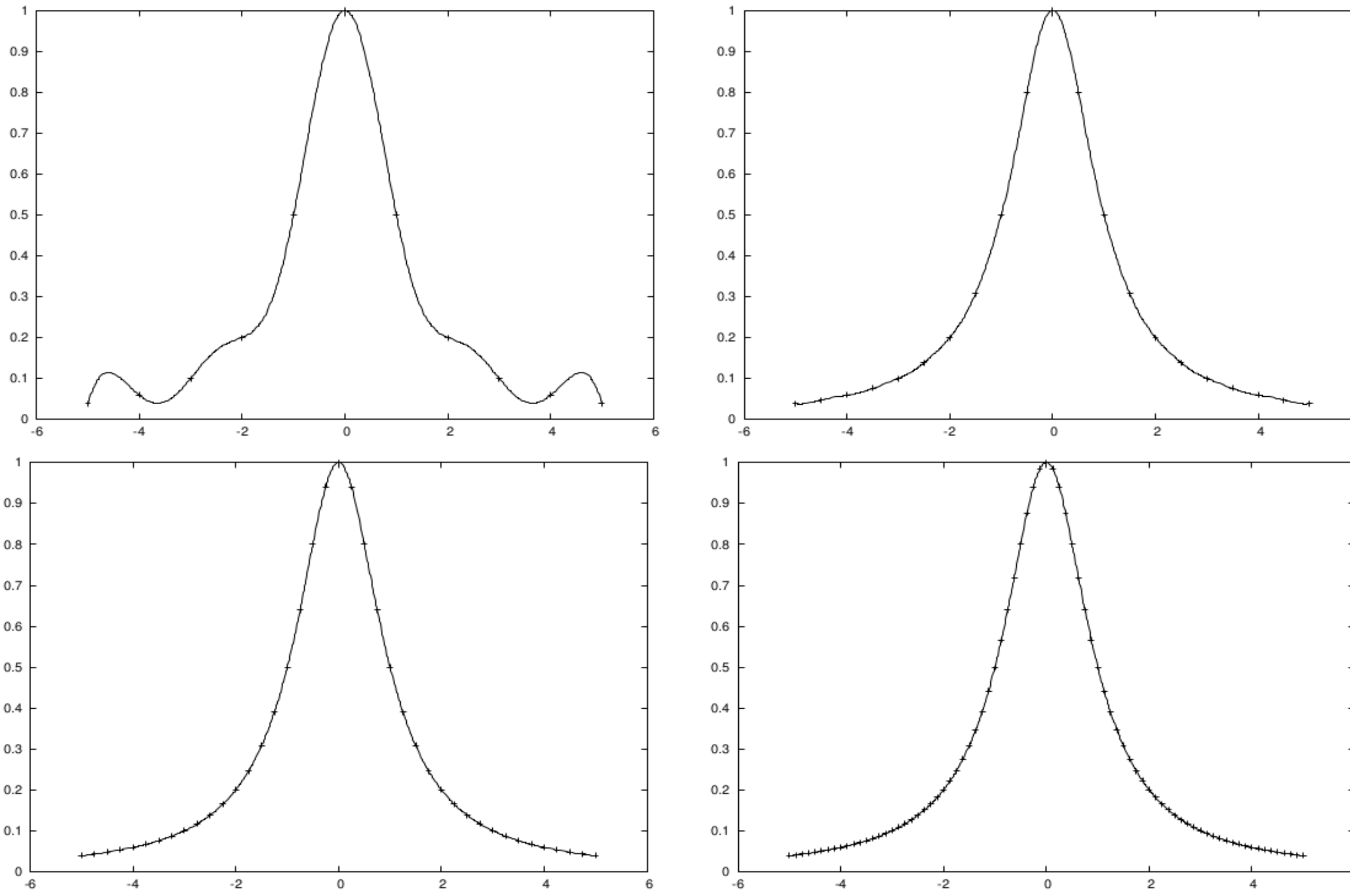


Figure 1: Interpolating Runge's example with $d = 3$ and $n = 10, 20, 40, 80$.

Sin(x) and errors for Sin(x) and Runge's Eq

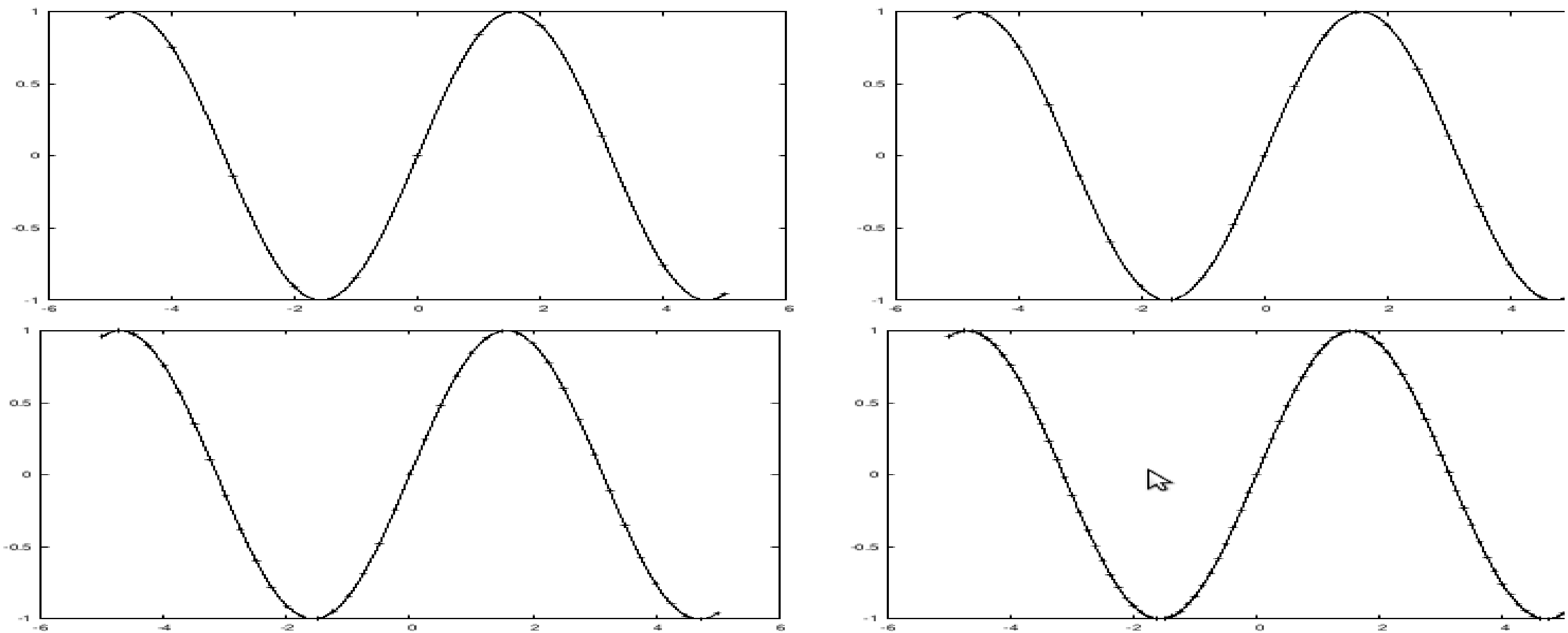


Figure 2: Interpolating the sine function with $d = 4$ and $n = 10, 20, 40, 80$.

n	Runge, $d = 3$	order	sine, $d = 4$	order
10	6.9e-02		1.7e-02	
20	2.8e-03	4.6	3.9e-04	5.5
40	4.3e-06	9.4	7.1e-06	5.8
80	5.1e-08	6.4	1.3e-07	5.7
160	3.0e-09	4.1	2.7e-09	5.6
320	1.8e-10	4.0	6.0e-11	5.5
640	1.1e-11	4.0	1.5e-12	5.3

Table 1: Error in rational interpolant.

Optimal Order for y = Abs(x)

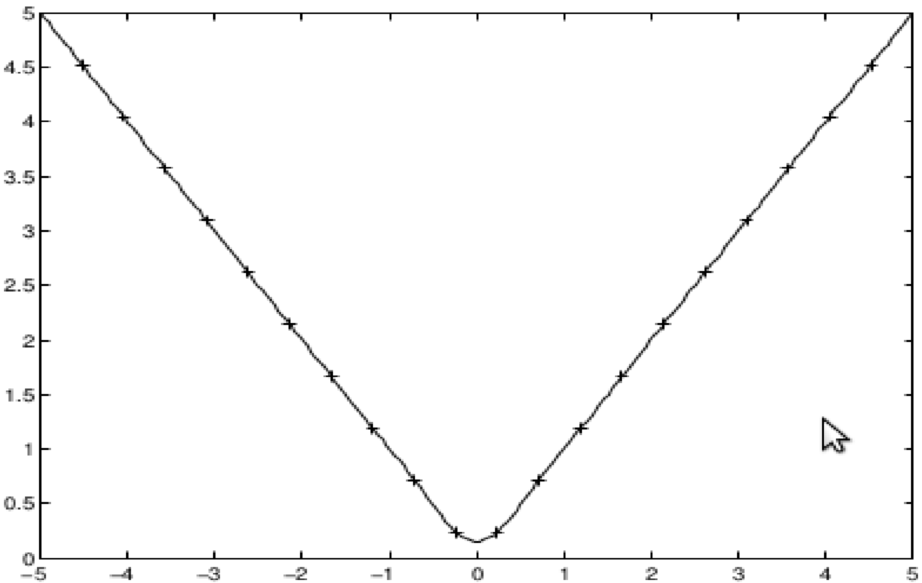


Figure 3: Interpolating $|x|$ over $[-5, 5]$ with $d = 3$ and $n = 21$.

n	best d value	error
10	$d = 0$	3.6e-02
20	$d = 1$	1.5e-03
40	$d = 3$	4.3e-06
80	$d = 7$	2.0e-10
160	$d = 10$	1.3e-15

Table 2: Error in Runge's example, varying d.

Errors in barycentric vs cubic spline interpolation for Runge's equation and the sine function

n	rational, $d = 3$	cubic spline
10	6.9e-02	2.2e-02
20	2.8e-03	3.2e-03
40	4.3e-06	2.8e-04
80	5.1e-08	1.6e-05
160	3.0e-09	9.5e-07
320	1.8e-10	5.9e-08
640	1.1e-11	3.7e-09

n	rational, $d = 3$	cubic spline
10	1.3e-02	3.3e-03
20	1.2e-03	1.7e-04
40	8.4e-05	1.0e-05
80	5.4e-06	6.4e-07
160	3.4e-07	4.0e-08
320	2.1e-08	2.5e-09
640	1.3e-09	1.6e-10

Table 3: Error in rational and spline interpolation of Runge's (left) and the sine function (right)

Rational Function Algorithm

$$R_{i(i+1)\dots(i+m)} = \frac{P_\mu(x)}{Q_\nu(x)} = \frac{p_0 + p_1 x + \cdots + p_\mu x^\mu}{q_0 + q_1 x + \cdots + q_\nu x^\nu} \qquad (3.4.1)$$

Since there are $\mu + \nu + 1$ unknown p's and q's (q_0 being arbitrary), we must have

$$m + 1 = \mu + \nu + 1 \qquad (3.4.2)$$

Recurrence Relations

Polynomial Approximations

$$P_{i(i+1)\dots(i+m)} = \frac{(x - x_{i+m})P_{i(i+1)\dots(i+m-1)} + (x_i - x)P_{(i+1)(i+2)\dots(i+m)}}{x_i - x_{i+m}} \tag{3.2.3}$$

Rational Function Approximations

$$R_{i(i+1)\dots(i+m)} = R_{(i+1)\dots(i+m)} + \frac{R_{(i+1)\dots(i+m)} - R_{i\dots(i+m-1)}}{\left(\frac{x-x_i}{x-x_{i+m}}\right)\left(1 - \frac{R_{(i+1)\dots(i+m)}-R_{i\dots(i+m-1)}}{R_{(i+1)\dots(i+m)}-R_{(i+1)\dots(i+m-1)}}\right) - 1} \tag{3.4.3}$$

Barycentric Algorithm

Barycentric form of rational interpolant

$$R(x) = \frac{\displaystyle\sum_{i=0}^{N-1} \frac{w_i}{x - x_i} y_i}{\displaystyle\sum_{i=0}^{N-1} \frac{w_i}{x - x_i}} \tag{3.4.9}$$

$$w_k = \sum_{\substack{i=k-d \\ 0 \le i < N-d}}^{k} (-1)^k \prod_{\substack{j=i \\ j \ne k}}^{i+d} \frac{1}{x_k - x_j} \tag{3.4.10}$$

Formula for the weights

For example,

$$\begin{aligned} w_k &= (-1)^k, & d &= 0 \\ w_k &= (-1)^{k-1}\left[\frac{1}{x_k - x_{k-1}} + \frac{1}{x_{k+1} - x_k}\right], & d &= 1 \end{aligned} \tag{3.4.11}$$

Coefficients of Interpolating Polynomials and Vandermonde Matrix

$$y = c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1} x^{N-1} \qquad (3.5.1)$$

e c_i's are required to satisfy the linear equation

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} \qquad (3.5.2)$$

- Transforms coefficients of a polynomial to the actual values it takes at particular points.
- Vandermonde determinant is non-vanishing for these points proving that the mapping is a one-to-one correspondence between coefficients and values . . . i.e. coefficients in polynomial interpolation have a unique solution

Coefficients continued

- Problems
  - Ill-conditioned as N increases . . . so technique only practical for small data sets
  - If coefficients are used to interpolate functions, the interpolation will not pass through data points
  - First algorithm in NR3 has $O(N^2)$; second has $O(N^3)$
  - For high degrees of interpolation, precision of coefficients is essential . . . so interpolation error compounded by inaccuracy of coefficients

Citations

http://www.alglib.net/interpolation/rational.php

http://www.alglib.net/interpolation/spline3.php

http://en.wikipedia.org/wiki/Linear_interpolation

http://en.wikipedia.org/wiki/Spline_interpolation

http://en.wikipedia.org/wiki/Polynomial_interpolation