Qualitative Direction Calculi with Arbitrary Granularity

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Abstract. Binary direction relations between points in two-dimensional space are the basis to any qualitative direction calculus. Previous calculi are only on a very low level of granularity. In this paper we propose a generalization of previous approaches which enables qualitative calculi with an arbitrary level of granularity. The resulting calculi are so powerful that they can even emulate a quantitative representation based on a coordinate system. We also propose a less powerful, purely qualitative version of the generalized calculus. We identify tractable subsets of the generalized calculus and describe some applications for which these calculi are useful.

1 Introduction

Spatial information is an important part of intelligent systems. There are mainly two different approaches to representing and reasoning about spatial information. One approach tries to represent spatial information in a quantitative, metric way, usually by some kind of coordinate system. Another approach, qualitative spatial representation and reasoning, tries to represent spatial information by specifying qualitative relationships between spatial entities. One of the main motivations of qualitative spatial representation is that it is considered to be similar to the way humans conceptualize spatial information and to the way it is expressed in natural language. An obvious advantage of qualitative spatial representation is the handling of imprecise information. Using qualitative relationships it is possible to express only as much information as is necessary or known. The level of precision which can be represented depends on the granularity of the qualitative relations.

Several aspects of space can be represented in a qualitative way, the most important being topology, direction, and distance [2]. In this paper we focus on qualitative direction, i.e., relationships such as left, right, north, or south. Three kinds of spatial entities are usually distinguished, points, lines or line segments, and extended two- or higher dimensional regions. While regions are certainly the most important spatial entities (real-world objects are three-dimensional extended regions) points are the most basic spatial entities and particularly important for representing directions. All qualitative direction calculi for one or higher dimensional spatial entities are essentially based on qualitative directions over points: line segments can be represented by the two end-points, lines by any two of their points, and direction relations between extended regions depend on certain points of the regions such as the leftmost point or the center of a region. Sometimes regions can even be approximated as points, in particular if the size of the regions is small compared to their distance. Therefore, developing a sophisticated qualitative direction calculus over points and exploring its limits is an essential part of qualitative spatial representation and reasoning.

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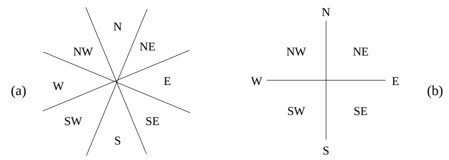

Fig. 1. Cone-based and projection-based qualitative direction calculi [4]

2 Representing Qualitative Direction

The direction between two spatial entities requires to specify either a reference point or a reference direction with respect to which the direction is measured, i.e., direction is essentially a ternary relation. In this paper we consider directions with respect to a given reference direction, i.e., we consider absolute and global directions. Under certain conditions relative directions can be transformed into absolute directions and *vice versa*, so this restriction is not as strong as it seems.

The direction of a point q with respect to another point p can be specified by using the exact angle between the vector $\boldsymbol{pq}$ and the vector which points from p to the given reference direction, where the angle is between $0°$ and $360°$. If the exact angle is not known or not important, which is the case whenever humans without a tool for the exact measurement of directions are involved, then it is better to have a qualitative representation of direction. The goal of a qualitative representation of the direction between points in two-dimensional space is to specify a limited number of relations such that each relation covers a part of the $360°$ range and all relations taken together cover the $360°$ range completely. If in addition the relations do not overlap, they form a jointly exhaustive and pairwise disjoint (JEPD) set of relations, called *basic relations*. The number of basic relations and the way in which they partition the $360°$ range depends on the application and on the required level of granularity. Since the range which has to be covered by the qualitative relations is fixed and well-defined, direction is perfectly suited for calculi with a varying level of granularity. If the given direction information is less precise than that of the basic relations, then the union of different possible basic relations can be used. Complete lack of information can be expressed as the union of all basic relations. Thus, a full set of qualitative relations contains all basic relations $\mathcal{B}$ and all possible unions of the basic relations $2^{\mathcal{B}}$. An essential requirement for applying standard qualitative reasoning algorithms is that the set of relations is closed under union, intersection, converse, and composition.

Frank [4] distinguished two kinds of calculi for representing absolute directions (see Fig. 1). The *projection-based approach* is based on two orthogonal axes where the four main directions north, east, south, and west are located on, and on the four sectors bounded by the axes which correspond to the directions northeast, southeast, southwest, and northwest. The *cone-based approach* is based on four axes which bound eight equally sized sectors corresponding to the eight above mentioned directions. The computational properties of the projection-based approach, also known as the *cardinal algebra*, has been analyzed by Ligozat [6]. Reasoning over the full cardinal algebra is NP-complete, while there exists a maximal tractable subset which contains all basic relations. For both approaches the granularity of the relations is fixed, which strongly limits their applicability.

3 The Star Calculus

In this section we introduce the *Star calculus* for representing and reasoning about qualitative directions between points in a two-dimensional space with respect to a given reference di-

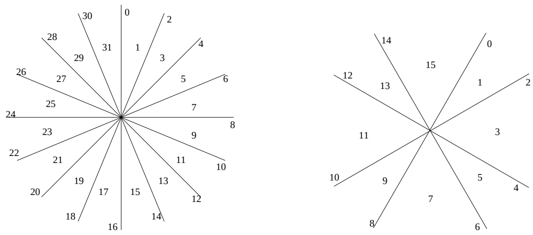

Fig. 2. Two Star calculi: (a) $\mathcal{STAR}_8(0)$, (b) $\mathcal{STAR}_4[30, 60, 120, 150](30)$

rection.[3] The Star calculus is a generalization of several existing calculi such as the different kinds of calculi which Frank distinguished. We give a general specification of a qualitative direction calculus which allows us to define basic relations on an arbitrary fixed level of granularity. We define converse and composition in a general way and prove general computational properties. Using the Star calculus, it is possible to specify qualitative direction relations between two points with respect to a given reference direction. For each point, the Star calculus divides the plane into several zones which form the different relations.

Definition 1 (Star calculus). *Given a two dimensional plane $\mathcal{P}$ and a global reference direction in $\mathcal{P}$. For each point $p \in \mathcal{P}$ the* Star calculus $\mathcal{STAR}_m[\delta_1, \dots, \delta_m](\delta_1)$ *where $0 \leq \delta_1 < \dots < \delta_m < 360$ and $\delta_m - \delta_1 < 180$ specifies m lines which intersect at p while forming the angles δ_j with the reference direction for each $1 \leq j \leq m$. For each point $p \in \mathcal{P}$ these m lines partition $\mathcal{P}$ into $4m + 1$ disjoint zones with respect to the reference direction (see Fig. 2): $2m$ half lines resulting from the m lines, $2m$ two-dimensional sectors each bound by two half lines, and the point p. A unique identifier is assigned to each zone as follows:*

- *The m lines are split into $2m$ half-lines which point from p to the directions δ_j and $(\delta_j + 180)$ mod 360 for $1 \leq j \leq m$. We assign the unique identifiers $0_p, 2_p, \dots, (4m-2)_p$ in clockwise order starting with the angle δ_1.*
- *The $2m$ half lines and the point p bound $2m$ sectors. We assign these sectors the unique identifiers $1_p, 3_p, \dots, (4m-1)_p$ in clockwise order starting with the sector bound by p and the half lines 0_p and 2_p.*

Using these zones, $4m + 1$ basic Star relations *can be defined as follows:*

1. *the identity relation $id \equiv \{(p, p) | p \in \mathcal{P}\}$, and*
2. *the relations $I \equiv \{(p, q) | p, q \in \mathcal{P}$ and $q \in I_p\}, \forall I \in \{0, 1, 2, \dots, 4m - 1\}$*
3. *we denote the relations $\{1, 3, 5, ...\}$ as* odd *relations, and $\{0, 2, 4, ...\}$ as* even *relations*

A Star calculus is the power set of all basic Star relations, i.e., it contains 2^{4m+1} different relations. The set of $4m + 1$ basic relations of a Star calculus $\mathcal{A}$ is denoted $bas(\mathcal{A})$. A Star calculus is called regular, *if all $2m$ zones have equal size, i.e., if the angle between consecutive lines is $180/m$ degrees, and can be written as $\mathcal{STAR}_m(\delta_i)$. The class of all Star calculi based on m lines is denoted $\mathcal{STAR}_m$.*

The union of different basic relations can be written as a set of basic relations, e.g., the union of the basic relations 2, 5, and 6 is written as $\{2, 5, 6\}$. The union of all basic relations, the *universal relation*, is written as $\{*\}$.

[3] The Star calculus was introduced in a slightly different form by Mitra [8], but in this paper we provide a rigorous formal defi nition and analysis of the calculus which results in surprisingly new insights.

Definition 2 (range). *If a union of relations covers a complete range of consecutive basic relations, e.g.* $\{2, 3, 4, 5, 6\}$*, we can write this as the range* $[2 - 6]$*. If the union contains the identity relation, we write* $+$ *instead of* $-$*, i.e.,* $\{2, 3, 4, 5, 6, id\}$ *can be written as the range* $[2 + 6]$*. A range is always written in clockwise order, i.e.,* $[2 - 6]$ *is different from* $[6 - 2]$ *(which represents* $\{6, \ldots, 4m - 1, 0, 1, 2\}$*). An* open range $]R - S[$ *or* $]R + S[$ *excludes the first and the last relation* R *and* S *from the range if they correspond to half-lines, i.e., if they are even numbers. E.g.,* $]2 - 6[=]1 - 5[= [1 - 5]$.

Let us specify the usual operators for the Star relations. Union and intersection of Star relations are the normal set-theoretic operators. Converse and composition depend on the semantics of the relations. Although a Star calculus can have sectors of different sizes, each relation has a definite converse relation because of the point symmetry of the lines intersecting at p.

Proposition 1 (converse). *Given a Star calculus* $\mathcal{A} \in \mathcal{STAR}_m$*. The converse of the relation* id *is* id*. The converse* $R^{\smile}$ *of a basic relation* $R \in bas(\mathcal{A})$ *with* $0 \leq R \leq 4m - 1$ *is given by* $(R + 2m)$ *mod* $4m$*. The converse of a union of basic relations is equal to the union of all converse basic relations. The converse of a range* $[R \pm S]$ *results in* $[R^{\smile} \pm S^{\smile}]$*. Analogous for open ranges.*

The notion of ranges is particularly helpful for specifying compositions of basic relations since these mostly cover consecutive relations. The composition operator is usually defined as $x(R \circ S)y =_{def} \exists z : xRz \wedge zSy$.

Proposition 2 (composition). *Given a Star calculus* $\mathcal{A} \in \mathcal{STAR}_m$*. Composition* $R \circ S$ *of two basic relations* $R, S \in bas(\mathcal{A})$ *can be computed as follows:*

1. If $R = id$*, then* $R \circ S = S$*,* *2. If* $S = id$*, then* $R \circ S = R$*,*
3. If $R = S$*, then* $R \circ S = R$*,* *4. If* $R = S^{\smile}$ *and* R *odd, then* $R \circ S = \{*\}$*,*
5. If $R = S^{\smile}$ *and* R *even, then* $R \circ S = \{R, S, id\}$*,*
6. Let X *be the shortest distance between* R *and* S*, i.e.,* $(R + X)$ *mod* $4m = S$*.*
If $X < 2m$*, then* $R \circ S =]R - S[$*. If* $X > 2m$*, then* $R \circ S =]S - R[$*.*

It follows from the above rules that $R \circ S = S \circ R$. Please note that the relations resulting from the last composition rule never contain boundary relations corresponding to half lines. This is because id is not contained in these relations, so either with R or with S one is always forced to leave the bounding lines. As usual, the composition of unions of basic relations is the union of the composition of each involved basic relation. It is surprising that the angles of the lines do not seem to be important, but we will see later that this is not always the case.

4 Reasoning over the Star calculus

Qualitative spatial reasoning consists of several different reasoning problems such as deriving unknown relations from a given set of spatial constraints or eliminating all impossible labels from given spatial constraints in order to obtain the minimal representation. Most of these problems can be reduced to the *consistency problem* CSPSAT($\mathcal{S}$) where $\mathcal{S} \subseteq 2^{\mathcal{B}}$ and $2^{\mathcal{B}}$ is closed under the usual operators [10]:

Instance: Given a set $\mathcal{V}$ of n variables over a domain $\mathcal{D}$ and a finite set Θ of binary constraints xRy where $R \in \mathcal{S}$ and $x, y \in \mathcal{V}$.
Question: Is there an instantiation of all n variables in Θ with values from $\mathcal{D}$ which satisfies all constraints?

For Star calculi, the domain $\mathcal{D}$ of variables is the set of all points $p \in \mathcal{P}$.

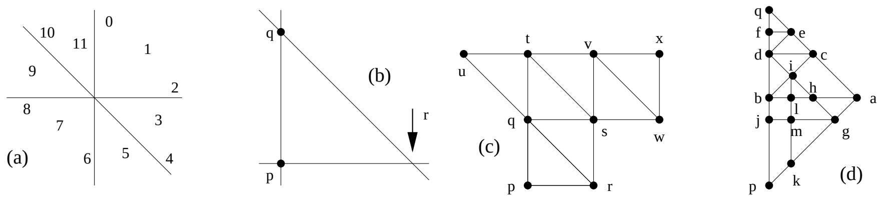

Fig. 3. Constructing the exact position of points with respect to p and q using lines with fi xed angles

Theorem 1. *Given a Star calculus $\mathcal{A} \in \mathcal{STAR}_m$. If $m = 1$, then* CSPSAT*($\mathcal{A}$) is tractable. If $m \geq 2$, then* CSPSAT*($\mathcal{A}$) is NP-complete.*

Proof Sketch. Let $m = 1$. We define a x-y-coordinate system in $\mathcal{P}$ where axis x is parallel to the line given by $\mathcal{A}$ and axis y is orthogonal to x. If we consider $p \in \mathcal{P}$ the center of the coordinate system, then the five zones defined by $\mathcal{A}$ with respect to p can be projected onto the two axes and be described as pairs of point algebra relations in the following way $\{\{>\}, \{=\}\}$, $\{\{<=>\}, \{<\}\}$, $\{\{<\}, \{=\}\}$, $\{\{<=>\}, \{>\}\}$, and $\{\{=\}, \{=\}\}$, where the first point relation specifies the projection onto the x axis and the second point relation the projection onto the y axis. This carries over to the Star relations which can be described in the same way. Any set of constraints Θ over $\mathcal{A}$ can be divided into two sets Θ_x and Θ_y of point algebra constraints as described above. First Θ_y is solved while eliminating all unforced equalities. Then all constraints of Θ_x referring to variables which are not equal in the solution of Θ_y are eliminated, resulting in Θ'_x. Θ is consistent iff Θ_y and Θ'_x are both consistent. For $m = 2$, NP-completeness follows from the NP-completeness result by Ligozat [6]. This carries over to all $m > 2$. ■

The next step is usually to study the computational properties of the basic relations, and, if reasoning over them is tractable, identify maximal tractable subsets. Before doing so, we first have a closer look at the expressiveness of the Star relations. As we will see, they are amazingly powerful as they actually allow us to express geometrical statements. Assume we have a Star calculus with three lines as show in Figure 3(a) and the three constraints $p\{0\}q, q\{4\}r, p\{2\}r$. If the constraints are consistent, then for all consistent instantiations of p and q, the instantiation of r is exactly determined as the intersection of the two lines 2_p and 4_q (see Figure 3(b)). This is an immediate consequence of Euclid's AAS and ASA theorems as the even Star relations have fixed angles. If we have two more constraints $q\{2\}s, r\{0\}s$, then the position of s is also exactly determined with respect to the instantiations of p and q, although the direct relation between p and s can only be expressed as $p\{1\}s$ which is a two-dimensional sector. In the same way we can continue and form an infinite grid of points which are all exactly determined with respect to the instantiations of two points p and q (see Figure 3(c)). If more than three lines are available, it is possible to exactly determine an infinite number of points between two other points of the grid (see Figure 3(d)), i.e., we can get coordinate system with rational values. This demonstrates that Star calculi with three or more lines are so expressive that it is even possible to define coordinate systems, an essentially quantitative entity. This fact is summarized in the following proposition.

Proposition 3 (coordinate systems based on Star relations). *Given a Star calculus $\mathcal{A} \in \mathcal{STAR}_m$ and two points $p, q \in \mathcal{P}$. If $m = 3$, it is possible to define a coordinate system with integer values with respect to two points p and q and the three given angles of $\mathcal{A}$ (see Figure 3(c)). If $m > 3$, it is possible to define a coordinate system with rational values with respect to two points p and q and the given angles of $\mathcal{A}$. (see Figure 3(d)).*

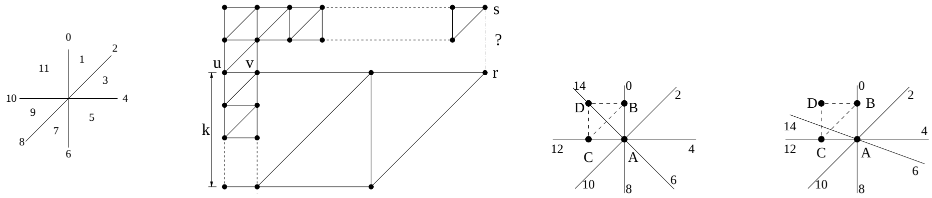

Fig. 4. (a) Construction for the proof of Theorem 2 (b) The Star calculi $\mathcal{A}$ and $\mathcal{B}$ of example 1

Thus, by using a Star calculus with more than two lines, the boundary between qualitative and quantitative representation has disappeared. The quantitative aspect of a coordinate system can be emulated by the Star calculus.

Since the exact positions of points which are enforced by the coordinate system cannot be determined by computing composition and intersection of other Star relations,it is quite obvious that this expressive power cannot be captured by qualitative reasoning methods based only on relational operations. A commonly used method of this kind is that of enforcing k-consistency [7]. A set of constraints Θ is called k-consistent, iff for *each* consistent instantiation of $k-1$ variables of Θ there is also a consistent instantiation of *any* k-th variable of Θ. The method of enforcing k-consistency consists in eliminating all basic relations of all tuples of k variables that contradict the results of applying all possible compositions within the tuple. Enforcing 3-consistency (also known as the *path-consistency method*), for instance, is done by computing for all triples of variables x, y, z of a set of constraints all entailments $R_{xz} := R_{xz} \cap (R_{xy} \circ R_{yz})$ until a fixed point is reached. If the empty relation occurs, the set is inconsistent. For $k \geq 3$ composition of relations of arity $k-1$ must be defined and used.

Theorem 2. *Given a Star calculus $\mathcal{A} \in \mathcal{STAR}_m$ with $m \geq 3$ and a constant $k \geq 2$. It is not possible to decide* CSPSAT*$(bas(\mathcal{A}))$ by enforcing k-consistency.*

Proof Sketch. Let $m = 3$. We use a Star calculus such as that in Fig. 4(a) (note that the lines could have arbitrary angles) and emulate a coordinate system with respect to two points u and v, i.e., we assume u has coordinate $(0,0)$ and v has coordinate $(1,0)$. We construct a point s with coordinate $(2k,2)$ by using a sequence of triangles with length 1. We further define a point r with coordinate $(2k+1,0)$ by using a sequence of triangles of length 1 followed by a sequence of triangles of length k (see Fig. 4(a)). Let Θ be the set of all constraints between all points involved in the construction where each constraint gives the exact basic relation between the points except for the relation between r and s which we set to $r\{0\}s$. In order to determine that Θ is inconsistent, it is necessary to map the points to a coordinate system or to count the relations which is impossible by trying to enforce k-consistency. Note that more than k variables are necessary in order to construct larger triangles of length k. For $m > 3$ an inconsistent set can be constructed in a similar way by making use of intermediate points on rational coordinates as shown in Figure 3(c). ∎

Note that the qualitative Star relations also depend on other geometrical laws such as the theorems of intersecting lines. The following example gives further indication that qualitative reasoning methods cannot be complete for Star calculi with $m \geq 3$.

Example 1. $\mathcal{A} = \mathcal{STAR}_4[0, 45, 90, 135](0)$ and $\mathcal{B} = \mathcal{STAR}_4[0, 45, 90, 110](0)$. Both calculi define 17 basic relations. Their composition and converse tables are identical. Let Θ be the set of constraints $A\{0\}B, A\{12\}C, C\{2\}B, C\{0\}D, B\{12\}D$ and let φ be the constraint $A\{14\}D$ (see Figure 4(b)). For both calculi Θ is consistent. $\Theta \cup \varphi$ is consistent for $\mathcal{A}$, but inconsistent for $\mathcal{B}$.

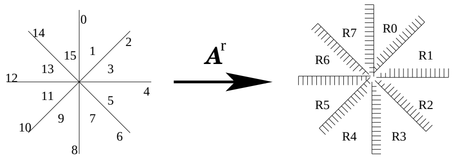

Fig. 5. The revised Star calculus $\mathcal{STAR}_4^r(0)$ obtained from $\mathcal{STAR}_4(0)$.

Despite the negative result of Theorem 2, CSPSAT($bas(\mathcal{A})$) is nevertheless tractable. It is just not possible to decide it with qualitative reasoning methods only, they only give approximate solutions to the consistency problem. It could be solved using quantitative methods such as expressing the constraints as a system of inequalities resulting from the algebraic semantics of the relations, or by combining qualitative reasoning methods with a coordinate system where intermediate results are entered, but this is outside the focus of this paper.

Since we are dealing with qualitative spatial representation and reasoning in this paper, we will in the following try to revise the Star calculus in a way which enables purely qualitative reasoning methods. In doing so, we will try to meet two important requirements for efficient qualitative reasoning over a particular calculus, namely, that tractable qualitative reasoning methods are sufficient for the set of basic relations of the calculus [9] and that all basic relations are contained in the tractable subsets of the calculus [11]. Since both requirements are not supported by the Star calculus, we have to define a revised version of the Star calculus which enables purely qualitative direction relations on arbitrary levels of granularity.

5 A Purely Qualitative Direction Calculus

Responsible for the high expressiveness of the Star calculus and for its ability to emulate a metric representation are the basic relations corresponding to the lines, i.e., the "*one-dimensional relations*". If these were eliminated or subsumed by new relations, it could be possible to obtain a sub-calculus of the Star calculus which allows for efficient qualitative reasoning. Removing the one-dimensional relations or adding them to both neighboring relations leads to relations which are not JEPD anymore. So we have to combine them with only one neighboring relation in order to obtain new basic relations. Since the resulting basic relations should be closed under converse, we have to combine them with either the preceding or the succeeding relations in clockwise order. We choose the first.

Definition 3 (revised Star calculus). *Given a Star calculus $\mathcal{A} \in \mathcal{STAR}_m$ for some $m \geq 2$ and its set of basic relations $bas(\mathcal{A}) = \{0, 1, \ldots, 4m-1, id\}$. The* revised Star calculus *$\mathcal{A}^r$ is the power set of the jointly exhaustive and pairwise disjoint set of relations $\mathcal{R} = \{R_0, R_1, \ldots, R_{2m-1}, id\}$ which is obtained from $bas(\mathcal{A})$ in the following way (see Figure 5): $R_i := \{2i+1, 2i+2\}$ for $0 \leq i \leq 2m-2$ and $R_{2m-1} := \{4m-1, 0\}$.*

The revised Star calculus consists of $2m+1$ basic relations. The converse of a basic relation R_i can be computed by $R_i{}^{\smile} = R_j$ where $j = i + m \bmod 2m$. While it is quite easy to guarantee that a system of relations is closed under the converse operator, it is much more difficult to show this for the composition operator. For the revised Star calculus the last composition rule of Prop. 2 is problematic, according to which the result of the composition of two basic relations is an open range. Therefore, the one-dimensional relations which are added to form basic relations of the revised Star calculus are removed again when computing composition, i.e., the revised Star calculus is not closed under composition. In order to fix this problem, we use *weak composition* [3], which is defined as the minimal relation which contains the actual composition, i.e., $R \circ_w S = \{T \mid T \cap (R \circ S) \neq \emptyset \text{ and } T \in bas(\mathcal{A})\}$. It is clear that the revised Star calculus is closed under converse, intersection and weak composition.

Proposition 4 (weak composition). *Given a revised Star calculus $\mathcal{A} \in \mathcal{STAR}^r_m$. The weak composition $R \circ_w S$ of two basic relations $R, S \in bas(\mathcal{A})$ can be computed as follows (again we have $R \circ_w S = S \circ_w R$):*

1. If $R = id$, then $R \circ_w S = S$, *2. If $S = id$, then $R \circ_w S = R$,*
3. If $R = S$, then $R \circ_w S = R$, *4. If $R = S^{\smile}$, then $R \circ_w S = \{*\}$,*
5. Let X be the shortest distance between R and S, i.e., $(R + X)$ mod $2m = S$. If $X < m$, then $R \circ_w S = [R - S]$. If $X > m$, then $R \circ_w S = [S - R]$.

For revised Star calculi weak composition differs from actual composition only for points located on the boundary of sectors of other points. If we can prove that for any consistent set of constraints, there exists a solution where no point lies on a sector boundary of other points, then weak composition can be used instead of composition for determining consistency.

Theorem 3. *Let Θ be a consistent set of constraints over a revised Star calculus $\mathcal{A}$. Then there is an instantiation of all variables in Θ with points of $\mathcal{P}$ such that no point is located on the boundary of a sector defined by another point.*

Proof Sketch. Let θ be a consistent instantiation of Θ. We can transform θ to an instantiation θ' such that no two points in θ' are located on each others sector bounding lines. We first compute the set of constraints Θ_θ over $bas(\mathcal{A})$ which hold between each pair of points in θ. Since Θ_θ is consistent, each point of θ lies in the intersection of the corresponding sectors of all other points. If the intersection forms an extended region and a point p is located at the boundary of it, then p is moved by $\epsilon > 0$ into the interior of the intersection. If the intersection is a line (segment), then there must be at least two points which are located on each others sector bounding line. If only two points are on the line, one can be moved by $\epsilon > 0$ into the interior of the others sector. If more than two points are on the line, the point which does not lie between two other points is moved first. In all cases, no two resulting points of θ' are located on each others bounding lines while all constraints are still satisfied. ∎

In order to determine tractable subsets of the revised Star calculus (the NP-completeness results of the Star calculus also hold for the revised calculus), we can make use of Helly's theorem [1]: "Let F be a finite family of at least $n+1$ convex sets in R^n such that every $n+1$ sets in F have a point in common. Then all the sets in F have a point in common." Applied to our case of two-dimensional space, it is necessary that any three convex sets have a point in common. Since any basic relation of the revised Star calculus (note: also any basic relation of the Star calculus!) is a convex set, any 4-consistent set of constraints Θ over the basic relations is consistent.[4] This is because of the definition of 4-consistency which states that for any consistent triple of variables of Θ a consistent instantiation of any fourth variable can be found. In this case the fourth point must be located in the intersection of the three sectors determined by the relations between each of the three points of the triple and the fourth point, which is the requirement for applying Helly's theorem. This holds for all "convex relations".

Definition 4 (convex relations). *Let $\mathcal{A}$ be a (revised) Star calculus. The set of* convex relations *$\mathcal{C}_\mathcal{A}$ consists of $\{*\}$ and of all relations of $\mathcal{A}$ which correspond to a range of consecutive basic relations which does not contain a basic relation* and *its converse: $\mathcal{C}_\mathcal{A} = \{[R \pm S] \mid R, S \in bas(\mathcal{A}),\ R^{\smile}, S^{\smile} \notin [R \pm S]\} \cup \{*\}$*

Theorem 4. *Any 4-consistent set of constraints Θ over the set of convex relations $\mathcal{C}_\mathcal{A}$ of a (revised) Star calculus $\mathcal{A}$ is consistent.*

[4] This is true for the Star calculus and for the revised Star calculus. In the following we write "(revised) Star calculus" if something applies to the Star calculus and to the revised Star calculus.

Since this theorem results from Helly's theorem, it is possible to compute an instantiation of Θ without backtracking by starting with three points and sequentially adding the other points in arbitrary order. For the Star calculus, this also applies to pre-convex relations (relations whose topological closure are convex relations) [5], while this concept is not applicable to the revised Star calculus. Unfortunately, this nice theorem is hardly useful for (revised) Star calculi $\mathcal{A} \in \mathcal{STAR}_m^{(r)}$ with $m \geq 3$. Not because it appears to contradict Theorem 2 (which it doesn't), but because 4-consistent sets are very rare and because it is not possible to enforce 4-consistency on every set of constraints over the (revised) Star relations, i.e., there are consistent sets of constraints for which there exists no corresponding 4-consistent set. Even for a revised Star calculus with $m = 3$, there are always consistent sets of constraints where not *all* consistent instantiations of three variables can be extended to a fourth variable: Consider the set of constraints involving the variables p, q, r, s in Figure 3(c). The set is clearly consistent, but when assigning $p = (0, 0)$, $q = (0, 1)$, and $s = (2, 1)$ it is not possible to find a consistent instantiation for r, hence the set is not 4-consistent and cannot be made 4-consistent. In any case, we are mainly interested in consistency, i.e., whether *at least one* consistent instantiation exists. 4-consistency is too restrictive for that purpose. It turns out that the path-consistency method (also known as enforcing 3-consistency, see Section 4) is sufficient in some cases.

Theorem 5. *Given a revised Star calculus $\mathcal{A} \in \mathcal{STAR}_m^r$ with $m \leq 3$. Consistency of a set of constraints Θ over $bas(\mathcal{A})$ can be decided by the path-consistency method.*

Proof Sketch. Proof by induction over the number of variables n of Θ. It clearly holds for $n = 3$. Assume (a) that it holds for $n = k$. Given a path-consistent set Θ with $n = k + 1$ variables $\mathcal{V} = \{v_1, \ldots, v_{k+1}\}$. The set $\Theta' \subset \Theta$ contains all constraints over the variables $\mathcal{V} \setminus \{v_{k+1}\}$, it is path-consistent and by assumption consistent. Θ is consistent if there is an instantiation θ' of Θ' such that all sectors of the points in θ' which are supposed to contain $\theta'(v_{k+1})$ have a non-empty intersection. Assume (b) that there is no such θ'. Because of Helly's theorem there must then be three variables v_i, v_j, v_l such that the corresponding sectors of $\theta'(v_i), \theta'(v_j), \theta'(v_l)$ do not have a common intersection. By a case analysis and a proof that the cases are the same for all possible angles of the given lines, we can show that for a Star calculus with $m \leq 3$ every path-consistent set of atomic constraints over four variables is always consistent, and, furthermore, that there is a subset of three variables such that for each consistant instantiation of the subset, there is always a consistent instantiation of the fourth variable. This fact (it is not because of assumption (a)) contradicts assumption (b). Therefore, there is a θ' which can be extended to a consistent instantiation θ of Θ. ■

6 Applications

All real-world objects are three-dimensional extended entities rather than points, so many applications of qualitative spatial direction relations require calculi developed for more complex spatial entities than points. For these calculi, the Star calculus will definitely be an important basis. Nevertheless, there are also quite some applications where direction relations between points are necessary. Whenever the distance between two extended spatial entities is large relative to their extension, it is more convenient to treat them as point-like entities. Navigation tasks consider almost always point-like entities which have to be navigated from one location to another. For vehicles with automatic navigation devices it is possible to follow an exactly specified direction. When using "normal" vehicles like a car or when hiking, then it is not possible to follow an exact direction. Depending on the navigation tool which is available (if any), a direction can only be specified within a certain angular range. This is where the Star calculus turns out to be useful. It allows to use the finest granulation which can be distinguished by the user. Navigation assistance such as a route description can then be given in terms of Star relations, e.g., "go towards direction R_1 until landmark A can be seen in direction R_2, then turn to direction R_3 and go until..." As briefly described in this example, Star calculi can also be used to locate positions relative to the directions of landmarks, which is one possible qualitative spatial reasoning task. This kind of navigation and reasoning is particularly useful in the open field. A similar task is to reason about positions and routes relative to (cell phone) transmitters. The cells formed by a transmitter cover zones with different angles which can be exactly represented by a Star calculus. Another application is the automatic recognition and interpretation of route sketches. Depending on the precision with which this recognition should be obtained, the granularity of the Star calculus can be chosen. For all these applications revised Star calculi can be used instead of full Star calculi.

7 Discussion

Qualitative directions between points are at the heart of qualitative spatial representation and reasoning. Directions are one of the most important spatial aspect and directions between points are the basis for any calculus over more complex entities. Moreover, there is also a number of applications where directions are required between points. In this paper we proposed a class of direction calculi, the Star calculi, which are a generalization of several existing calculi for representing and reasoning about the qualitative direction between points in a plane. Although it is an important property of qualitative representations that the qualitative relations can be chosen on a level of granularity which is useful for a certain application, previous approaches did not offer the possibility of selecting basic relations on the desired level of granularity. Star calculi offer this possibility and they do so in an unrestricted way. Basic direction relations can be chosen on an arbitrary level of granularity. Star calculi can therefore be adopted to applications in an optimal way. We give general rules for computing composition and converse for the whole class of calculi. This enables us to apply the well-known qualitative reasoning methods. It turns out that Star calculi are so powerful that it is even possible to emulate coordinate systems. The disadvantage of this expressiveness is that qualitative reasoning methods cannot be complete for most Star calculi, but only provide approximate solutions to the reasoning problems. Developing complete reasoning algorithms is a matter of future research. We proposed a less expressive version, the revised Star calculi which do not contain relations corresponding to lines but only to two-dimensional sectors. We analyzed the computational properties and identified tractable subsets for certain classes of the revised calculus. Future work on revised Star calculi should analyze other classes, identify maximal tractable subsets and develop algorithms for finding consistent instantiations.

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