Explanation Generation over Temporal Interval Algebra

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Abstract

Temporal interval algebra has generated strong interest for both theoretical and practical reasons. All its **maximal tractable subalgebras (MTS)** have been identified. Now is the time to make transition toward their practical applications. In this work we have proposed a formalism on how to classify an input temporal network in one of these MTSs, or decide its intractability. We have also proposed a linear algorithm for checking consistency when the input belongs to one of the seventeen MTSs, and for finding out the constraints responsible for inconsistency in case the network is unsatisfiable.

1. Introduction

Interval Algebra (IA) is possibly the most studied algebra related to automated reasoning, for its theoretical elegance and feasible practical applications in scheduling, natural language engineering, etc. Some of the hallmark works in the area are: Vilain and Kautz's [1986] proof of NP-hardness of IA; Nebel and Bürckert's [1995] detecting the first maximal tractable algebra, namely, the ORD-Horn algebra (OH); Ligozat's [1996] reinterpretation of that algebra in a canonical and geometrical representation space; and Krokhin et al.'s [2003] discovery of the exhaustive set of eighteen non-trivial Maximal Tractable Subalgebras (MTS). Sometimes we will use the word "algebra" or "subalgebra" synonymously with MTS.

A motivation behind studying the maximal tractable algebras is that an application domain may fall into one of these classes, or may be restricted to one of these classes making temporal reasoning more practical for that application. Given such an expectation it is only reasonable to ask, how can we identify an input system of interval constraints whether it belongs to any of the MTSs? In this paper we develop a classification structure of the MTSs and we propose an algorithm for identifying an input network if it belongs to any particular MTS. We use a novel geometrical interpretation of the MTSs for this purpose. Although, polynomial-time Path Consistency algorithm (PC) is complete for any MTS, we show that each MTS, other than the OH, has a similar behavior as the point-based reasoning problem (van Beek, 1992, Drakengren et al., 1997), thus, enabling one to apply a more efficient cycle-checking algorithm than the PC. This new algorithm also has an extension for detecting *culprit* constraints when the input is inconsistent. *Culprit* detection is equivalent to the diagnosis as a task.

In the following section we provide some background information on IA for uninitiated readers. Subsequently we will show the geometrical interpretation of the MTSs in Ligozat's canonical space. We will then introduce the classification algorithm for an arbitrary interval constraint network. Lastly we provide the scheme for checking consistency of some **Qualitative Temporal Constraint Networks (QTCNs)** belonging to any of the MTSs, other than the OH.

2. Background on Temporal Reasoning

Qualitative reasoning with intervals involves thirteen atomic relations, B: {*before(p), after(*p^{-1}*), meets(m), met-by(*m^{-1}*), overlaps(o), overlapped-by(*o^{-1}*), starts(s), started-by(*s^{-1}*), during(d), contains(*d^{-1}*), finishes(f), finished-by (*f^{-1}*), equal(eq)*}, between any pair of intervals [Allen, 1983]. The corresponding **relational algebra** is comprised of the power set $P(B)$, the power set of B, which is closed under the traditional reasoning operators like composition, converse, set union, and set intersection.

Definition 1: A *Qualitative Temporal Constraint Network* (QTCN) is a graph $G=(V, E)$, where each node denotes an interval, and each directed labeled edge $(v_1, v_2, R) \in E$ represents disjunctive constraint R between v_1 and v_2, where $R \in P(B)$. Two special relations are *tautology* (disjunction of all thirteen atomic relations or no constraint), and Ø (empty relation leading to inconsistency). Reasoning may be restricted to a subset Θ, where $R \in \Theta \subseteq P(B)$, in case Θ is closed under composition, converse and intersection, thus, forming a Θ-subalgebra.

*Definition 2: **Qualitative Temporal Reasoning Problem*** (QTR(Θ)) is to answer, given a QTCN Q in which only relations form Θ occur, if a satisfiable assignment for each of the nodes exists, such that all the constraints R in Q are satisfied. For Θ

= $P(B)$, the full algebra is called the *Interval Algebra* or *IA*. The reasoning problem over full *IA* is known to be NP-hard (Vilain and Kautz, 1986). For $\Theta \subset P(B)$, restricted reasoning is interesting if such a Θ-subalgebra is tractable.
Definition 3 (Maximal tractable subalgebra, or *MTS*): A tractable subalgebra Θ is maximal if it has no super-algebra Θ' that is tractable, other than the full algebra $P(B)$.
Eighteen MTSs for the *IA* have been found such that the list is exhaustive – no other MTS of *IA* exists [Theorem 2.3 of Krokhin et al. 2003]. This is true at least from linguistic perspective, i.e., we do not know if the graph structure of temporal network may provide more MTS. The list includes **ORD-Horn MTS**, which is the only MTS that includes all thirteen atomic relations in *B*. We call the set of MTSs other than OH as the *Krokhin-MTS.*

Ligozat [1996] developed a canonical way of representing time interval relations geometrically, which appears as a useful tool for understanding the MTSs (Fig 1): An intervals is placed as a point in a 2D-Cartesian space, where the starting point of the interval is *X*-coordinate, the ending point is *Y*-coordinate, and the valid space is *Y*>*X*, forbidding the interval ending point to occur before the starting point. For instance, if interval *A* '*precedes*' interval *B*, then *A* will be located as a point anywhere in the angular open space '*p*' while *B* is at the position designated as '≡.' The topological relationships between the regions corresponding to the atomic relations in this space constitute a lattice (Fig 1b, "*p*" as inferior (0,0) and "p^{-1}" as superior (4,4)).

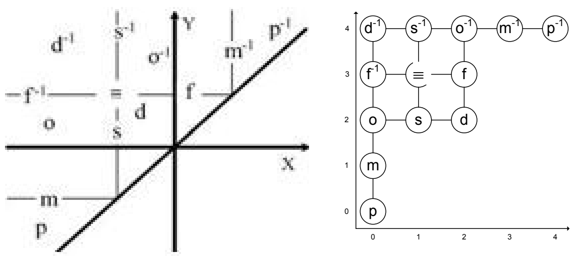


Figure 1a and 1b: Canonical representation of interval relations

We need the following definitions from Ligozat [1996].
Definition 4 (Convex relation): A *convex* relation is a closed lattice-interval on the lattice in Fig 1b.
Example 1: An interval between (0,2) and (2,4) in the lattice, which is a disjunctive set {*o, s, d,* f^{-1}, ≡, *f,* d^{-1}, s^{-1}, o^{-1}} is a convex relation.
Example 2: An interval from (0,3) to (2,3): { f^{1}, ≡, *f*}.
Definition 5 (Convex closure *Cl*(*r*)): Convex closure of a relation *p* is the disjunction of all atomic relations within the minimum lattice-interval over the lattice enclosing *r*.
Example 3: *Cl*({*o,* d^{-1}, ≡}) = {*o,* f^{1}, d^{-1}, *s,* ≡, s^{-1}}, or the range (0,2) through (1,4).
Definition 6 (Dimension *dim*(*l*)): For an atomic relation *b*, *dim*(*b*) is the dimension of *b* in the Canonical representation in Fig 1a. For any relation *r*, *dim*(*r*) = max{*dim*(*b*) | $b \in r$}.
Example 4: "*p*" is adjacent to "*m,*" dim(*p*)=2 and dim(*m*)=1.
Definition 7 (Preconvex relation): A preconvex relation *l* is such that $dim(cl(l) \setminus l) < dim(l)$. In other words, missing atomic relations from *cl*(*l*) are of relatively lower dimensions.
Example 5: {*o, s, d,* d^{-1}, o^{-1}} is a preconvex relation, where the missing relations are (*f,* ≡, f^{1}, s^{-1}) in order to form the respective convex closure. The convex closure being {*o, s, d, f,* ≡, f^{1}, d^{-1}, s^{-1}, o^{-1}} with the range from (0,2) to (2, 4) on the lattice.
Example 6: { f^{1}, *f*} is another preconvex relation. Missing lower dimensional relation is (≡) from the convex closure. The *Cl*({ f^{1}, *f*}) is { f^{1}, ≡, *f*} with the range from (0, 3) to (2, 3).
ORD-Horn relations are the same as the *preconvex* relations [Ligozat, 1996]. Set of ORD-Horn relations form a MTS (OH) of *IA*.

3. Geometrical representation of MTSs

The following Figures (Fig. 2(a), (b), (c), (d)) visualize the seventeen MTSs.

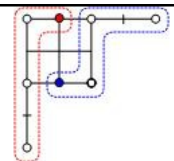
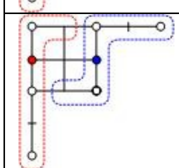
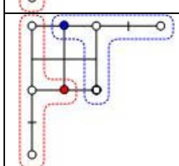
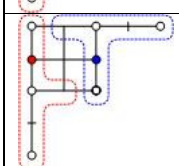
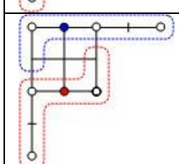
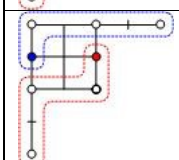
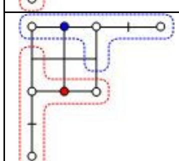
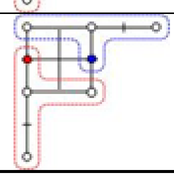
	Alj[1] or *A1* [Krohkin et al., 2003] $\{r \mid r \cap (pmod^{-1}f^{-1})^{\pm 1} \neq \emptyset \Rightarrow (s^{-1})^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[2] or *B3* $\{r \mid r \cap (pmod^{-1}s^{-1})^{\pm 1} \neq \emptyset \Rightarrow (f^{-1})^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[3] or *A2* $\{r \mid r \cap (pmod^{-1}f^{-1})^{\pm 1} \neq \emptyset \Rightarrow (s)^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[4] or *B4* $\{r \mid r \cap (pmod^{-1}s^{-1})^{\pm 1} \neq \emptyset \Rightarrow (f^{-1})^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[5] or *A3* $\{r \mid r \cap (pmodf)^{\pm 1} \neq \emptyset \Rightarrow (s)^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[6] or *B2* $\{r \mid r \cap (pmods)^{\pm 1} \neq \emptyset \Rightarrow (f)^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[7] or *A4* $\{r \mid r \cap (pmodf^{-1})^{\pm 1} \neq \emptyset \Rightarrow (s)^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$
	Alj[8] or *B1* $\{r \mid r \cap (pmods)^{\pm 1} \neq \emptyset \Rightarrow (f^{-1})^{\pm 1} \subseteq r\}$ $2^{11}+2^{6}+2^{6}+2 = 2178$

Figure 2a: Lattice representation of MTSs Alj 1-8

As described in the last section Ligozat [1996] provided the geometrical representation of the OH MTS. In this work we provide similar representation of the other seventeen non-trivial MTSs on the same canonical lattice over the atomic relations. Each MTS (except Alj[17]) M splits the lattice into up to four partitions (for the lack of any better term we will call them as "sublattices," although mathematically that is not necessarily accurate): sublattices $L1$, possibly $L2$ and $L3$, and a set of "free" atomic relations F (explained shortly, see Fig. 2a). In each sublattice we call a specific atomic relation as "pivot," pv_i (i=1, 2, and possibly 3). Any disjunctive relation $R \in M$ has the following property: if $R \cap L_i \neq \emptyset$, then $pv_i \in R$, i.e., the corresponding pivot to the lattice must be in R. Any atomic relation belonging to the "free" set F (not in any sublattice) may be in R. Alj[17] has the whole lattice as the only sublattice and $\equiv$ as the pivot. Presence of respective pivots is imperative for a relation to be in an MTS. Sublattices and pivots characterize each MTS. The left column in the figures (Fig 2) contains the lattice representation and the right column shows the original Krokhin et al.'s definition along with the counting of the algebra elements.

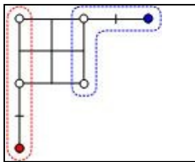
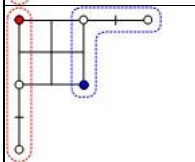
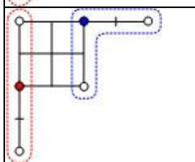
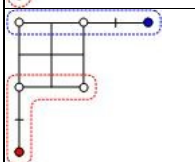
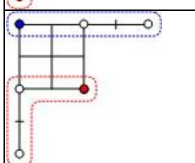
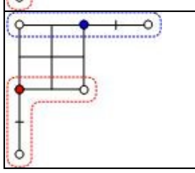
	Alj[9] or *Sp* $\{r \mid r \cap (pmod^{-1}f^{-1})^{\pm1} \neq \emptyset \Rightarrow (p)^{\pm1} \subseteq r\}$ $2^{11}+2^{7}+2^{7}+2^{3} = 2312$
	Alj[10] or *Sd* $\{r \mid r \cap (pmod^{-1}f^{-1})^{\pm1} \neq \emptyset \Rightarrow (d^{-1})^{\pm1} \subseteq r\}$ $2^{11}+2^{7}+2^{7}+2^{3} = 2312$
	Alj[11] or *So* $\{r \mid r \cap (pmod^{-1}f^{-1})^{\pm1} \neq \emptyset \Rightarrow (o)^{\pm1} \subseteq r\}$ $2^{11}+2^{7}+2^{7}+2^{3} = 2312$
	Alj[12] or *Ep* $\{r \mid r \cap (pmods)^{\pm1} \neq \emptyset \Rightarrow (p)^{\pm1} \subseteq r\}$ $2^{11}+2^{7}+2^{7}+2^{3} = 2312$
	Alj[13] or *Ed* $\{r \mid r \cap (pmods)^{\pm1} \neq \emptyset \Rightarrow (d)^{\pm1} \subseteq r\}$ $2^{11}+2^{7}+2^{7}+2^{3} = 2312$
	Alj[14] or *Eo* $\{r \mid r \cap (pmods)^{\pm1} \neq \emptyset \Rightarrow (o)^{\pm1} \subseteq r\}$ $2^{11}+2^{7}+2^{7}+2^{3} = 2312$

Figure 2b: Lattice representation of MTSs Alj 9-14

The lattice representation shows the corresponding sublattices enclosed by dashed lines, and the pivots as dark circles. Each algebra in Fig 2a (Alj 1 through 8) has

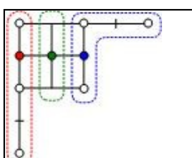
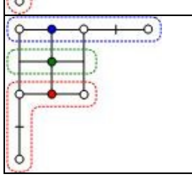
	Alj[15] or *S** $\{r \mid r \cap (pmod^{-1}f^{-1})^{\pm1} \neq \emptyset \Rightarrow (f)^{\pm1} \subseteq r$, and $r \cap (ss^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r\}$ $2^{10}+2^{8}+2^{6}+2^{6}+2^{4}+2^{4}+2^{2}+2^{0}=1445$
	Alj[16] or *E** $\{r \mid r \cap (pmods)^{\pm1} \neq \emptyset \Rightarrow (s)^{\pm1} \subseteq r$, and r $\cap (ff^{-1}) \neq \emptyset \Rightarrow (\equiv) \subseteq r\}$ $2^{10}+2^{8}+2^{6}+2^{6}+2^{4}+2^{4}+2^{2}+2^{0}=1445$

Figure 2c: Lattice representation of MTSs Alj 15-16

one dimensional pivots ($s^{\pm1}$ and $f^{\pm1}$), and $\equiv$ as the only free atom, in Fig 2b (Alj 9–14) has two-dimensional pivots and three free atoms each, in Fig 2c (Alj 15-16) has three lattices, lower dimensional pivots ($s^{\pm1}$, $f^{\pm1}$ and $\equiv$) and no free atom, and in Fig 2d the trivial "equality" Alj[17] has no free atom, the whole lattice is the only sublattice and the pivot is the $\equiv$. Note that OH cannot be visualized with a static depiction over the lattice as these seventeen MTSs could be.

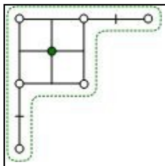
	Alj[17] or *A* $\{r \mid r \neq \emptyset \Rightarrow (\equiv) \subseteq r$ 2^{12}=4096

Figure 2d: Lattice representation of the MTS Alj 17

We will call OH algebra as Alj[18], which has no static geometrical representation as those of the Krokhin algebras.

4. MTS classification

As one can clearly see from the above discussion - there are only a few lattices and pivots that together comprise the seventeen MTSs. In Table 1 we tabulate the algebras according to these criteria. The table clearly visualizes the clustering of the algebras. Algebras appearing in a cluster have many similar properties, e.g., the number of elements.

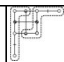
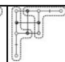
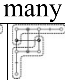
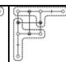
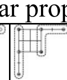
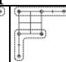
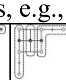
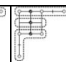
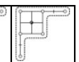
Sub-lattice struct → Pivots ↓									
$(s)(s^{-1})$	Alj[1]	Alj[3]	Alj[5]	Alj[7]					
$(f)(f^{-1})$	Alj[2]	Alj[4]	Alj[6]	Alj[8]					
$(p)(p^{-1})$					Alj[9]	Alj[12]			
$(d)(d^{-1})$					Alj[10]	Alj[13]			
$(o)(o^{-1})$					Alj[11]	Alj[14]			
$(f)(\equiv)(f^{-1})$							Alj[15]		
$(s)(\equiv)(s^{-1})$								Alj[16]	
$(\equiv)$									Alj[17]

Table 1: Algebra clusters

A given QTCN *C* belongs to an algebra *A* if each constraint *R* ($R \in C$) belongs to *A*. A QTCN *C* may belong to multiple MTSs, or none - in case the input is intractable. For this reason we initially presume that *C* belongs to all the algebras (step 1 of the algorithm in Fig 3). For any atomic relation *r* to belong to a relation *R* in an algebra, the algebra demands a strict inclusion of the corresponding pivot *pv* in the relation *R* unless *r* is a free atom (in *F*). This information led us to develop a 13x13 table (*AljTable* as Table 2) over the set of atomic relations (*B*). The rows indicate the atomic relations (*r*), columns indicate the absence of the pivot (*!pv*), and an entry in the *AljTable* indicates the algebras that the relation *R* may not belong to if $r \in R$ and $pv \notin R$. (See example 7 in section 5). Only first seven of the thirteen columns are shown in the table *2*, the rest of the entries can be derived from the symmetry. For example, Alj[*o*][$!f^{-1}$] = Alj[o^{-1}][*!f*].

	!b	!m	!o	!s	!d	!f	!≡
b			11,14	3,5,7,16	13	6	17
m	9,12		11,14	3,5,7,16	13	6	17
o	9,12			3,5,7,16	13	6	17
s	12		14		13	2,6	15,17
d	12		14	1,5,7,16		2,4,6,15	17
f				1,5	10		16,17
≡							
$\mathbf{f^{-1}}$	9		11	3,7			16,17
$\mathbf{d^{-1}}$	9		11	3		8	17
$\mathbf{s^{-1}}$						4,8	15,17
$\mathbf{o^{-1}}$				1	10	2,4,8,15	17
$\mathbf{m^{-1}}$				1	10	2,4,8,15	17
$\mathbf{b^{-1}}$				1	10	2,4,8,15	17

Table 2: AljTable[][]

Lines 2-5 of the algorithm ***FindMTSs*** excludes all the seventeen **Krokhin-MTSs** that the input QTCN *C* cannot belong to. A relation *R* belonging to OH algebra may not have any dimension 2 atomic relation in *cl(R)\R,* where *cl(R)* is the convex closure of *R*. Lines 6-10 checks for that. The *X* and *Y* coordinates on the lattice (Fig 1) of an atomic relation *r*, namely the *r(X)* and *r(Y)* are used for this purpose. The inverse function *rel(X, Y)* indicates the corresponding atomic relation on the lattice. If dimension 2 atomic relation is found in *cl(R)\R, C* does not belong to the ORD-Horn MTS, and is removed from the *Algebras* set (line 10). Finally, if *C* does not belong to any MTS, then *C* is an intractable problem (line 11). Fpr a proof of the intractability, see Krokhin et al.[2003].

Proposition 1: An entry in Table 2, *AljTable[x][!y]*, correctly shows the respective MTS to be excluded if any relation *R* in a TCN has an atomic relation *x*, but not the corresponding pivot *y*.

This is easy to verify from Table 1. For example, any relation R in Alj[11] may not have 'p' (from left vertical sublattice), while the corresponding pivot 'o' is absent from R (see Table 1).

Theorem 1: *Algorithm FindMTSs identifies a TCN C with the respective MTSs, or declares C to be intractable, in O(|C|) time, where |C| = the number of non-trivial relations in C (not tautology, or not Ø).*

Proof: Algorithm utilizes the defining characteristics (proposition 1) of the MTSs. The two *for-loops* on lines 3 and 8 are bounded by constant integers. Hence, the complexity from the loop on line 2 is O(|*C*|).

```
Algorithm FindMTSs
  Input: A QTCN C
  Returns: Find the set of algebras C belongs to, or
"intractable"
(1)  Initialize Algebras = {all eighteen MTSs};
(2)  ∀ relations R ∈ C do {
      // check for the  seventeen Krokhin-MTSs
(3)        ∀ atomic relation r, p ∈ B do
(4)            if r ∈ R && p ∉ R then
(5)                 Algebras := Algebras\ AljTable[r][!p];
       // check for the OH MTS
(6)      (lowX, lowY) := Min {r(X), r(Y) | r ∈ R };
(7)     (highX, highY) := Max {r(X), r(Y) | r ∈ R };
         // computed cl(R),  the convex closure of R
(8)      for (X:=lowX, Y:=lowY;
                      X ≤ highX, Y ≤ highY; X++, Y++) do
(9)         if rel(X, Y) ∉ R  && dim(rel(X, Y)) = = 2 then
        // cl(R)\R has a higher dimensional atomic relation
(10)           Algebras := Algebras\ Alj[18]; };
(11)  if Algebras == ∅ then return "intractable";
```

Figure 3: Algorithm FindMTSs

5. Consistency checking and Culprit detection for seventeen Krokhin-MTSs

Drakengren et al. [1997] suggested an algorithm to solve 21 large tractable subclasses of **Allen's Interval algebra**. The algorithm practically converts the input QTCN to a directed graph and looks for cycles to detect inconsistency. In that work, only nine of the algebras were identified as maximal. Those are the algebras we named Alj[i] for $1 \leq i \leq 8$ (Fig. 2a) and the OH algebra (ORD-Horn [Nebel and Bürckert, 1995]). The remaining 12 algebras were actually subalgebras of Alj[j] for $9 \leq j \leq 14$ mentioned in Figure 2b. For those algebras, excluding the ORD-Horn Algebra, Drakengren et al. [1997] suggested a polynomial time algorithm, which runs in linear time in the number of intervals input.

In this section, we first modify the existing algorithm to fit our terminologies, and enhance the former in two ways, (1) make sure that it is applied over all the Krokhin-MTSs, (2) suggest how the algorithm may be extended with new development toward OH MTS, so that one can com up with a complete algorithm for all MTS, and (3) store the constraints responsible for inconsistency (*culprits*) when the system is not satisfiable [Algorithm *SAT-TR* in Fig. 4]. To repeat, this algorithm can be applied to the remaining algebras (than the ones for which Drakengren et al. [1997] developed their algorithm for) presented in Figs. 2c and 2d, and for OH algebra. Identifying culprits is also a new direction in this line of work.

For the purpose of OH algebra, the *Proposition 4.8* of Drakengren et al. [1995] may be used:

Proposition 2. The only *maximal acyclic relations* (MAR) in *IA* are *(b d o m s f), (b d o m s* f^{-1}*), (b* d^{-1} *o m s* f^{-1}*), and (b* d^{-1} *o m* s^{-1} f^{-1}), and their respective converses.

An acyclic relation is such that *iff* all the arcs in a QTCN are labeled with a particular acyclic relation or its subsets, then the presence of a cycle in the QTCN indicates inconsistency. Maximal acyclic relation is maximal for all such acyclic relations in the usual sense of maximality. Let us call *(b d o m s f), (b d o m s* f^{1}*), (b* d^{-1} *o m s* f^{1}*), or (b* d^{-1} *o m* s^{-1} f^{1}*)* as a Forward MAR or *FMAR*, and an inverse of an FMAR as Backward MAR or *BMAR*.

```
Algorithm SAT-TR
 Input: a temporal constraint network C over θ-subalgebra, where θ∈(Alj[i], 1≤i≤ 18)
 Output: accept, or a set of culprit constraints
(1)   let C' := C, Culprit := ∅;
(1)   if i<18 in the input Alj[i]   // a Krohin-MTS
(1)     let pv be the forward non-zero dimensional pivot of Alj[i]
(3)       ∀ constraints (n1R n2)  ∈ C do
(4)         if pv ∉ R and pv^-1 ∈ R do
(5)             redirect (n1R n2)  to (n2 R^-1 n1);
(6)         if pv ∈ R and pv^-1 ∈ R do
(7)             C' := C' – R;


(8)     SC := the set of all nodes of all strong components in C';
(9)     ∀ constraints (n1R n2) ∈ C  do {
(10)        if R  ∩ {≡} == ∅  do
(11)          if n1 and  n2 ∈ SC do
(12)              Culprit := Culprit ∪ (n1R n2);
(13)        if R  == ∅
(14)          Culprit := Culprit ∪ (n1R n2)};
(15)  if Culprit == ∅    return 'accept';
(16)  else return Culprit
```

Figure 4: Algorithm SAT-TR

Note that a *culprit* edge may exist in two different strong components or may belong to the same strong component (line 12). This is why we collect all nodes belonging to the strong components in *SC* in line 8, which are the only important information for our purpose.

Theorem 2: *Algorithm **SAT-TR** correctly solves satisfiability for Alj[i] for* $1 \le i \le 17$.
Proof: Drakengren & Jonsson's [1997] proof is trivially extendable for all the first 14 MTS. *Alj[15]* and *Alj[16]* has three sublattices each (Fig. 2c), and the proof relies on the correctness of the theorem for some of those algebras. First, we must assume that no edge is labeled with ∅, otherwise, the problem is not satisfiable. For *Alj[15]*, it is easy to see that if no relation from the middle sublattice including *(≡)* is present, then the problem can be solved using *Alj[1]* or *Alj[3]*. If on the contrary, the middle sublattice is present, then so is the *(≡)* relation on every concerned labeled edge that is the pivot for middle lattices. Therefore, the output set '*Culprit*' will not be populated on this iteration, since the conditional statement in line 13 is not true, as long as the relation contains the *(≡)* relation. Similarly for *Alj[16]*, the two algebras that can be used are *Alj[5]* or *Alj[7]*. For *Alj[17]* every relation must include the *(≡)* and the problem is satisfiable for any input (as long as $\varnothing \notin C$).

Proof for *Alj[18]* (OH) is somewhat tricky for absence of any fixed pivots. Thus, the arcs here may not be directed as in line 5. Also, even if in some cases MAR's are used for directing the arcs the existence of the strong connected components are not sufficient condition for unstaisfiability of a network. This is because '*equality*' and '*inequality*' of the end points may for an inconsistent cycle. For example, *Interval1 {f,* f^{-1}*} Interval2 {s,* s^{-1}*} Interval3 {≡} Interval1*, no directed cycle exists in this case, but '*equality*' relations over the end points of the respective intervals conflict.

Theorem 3: *Algorithm SAT-TR runs in linear time in the size of C.*
Proof: See Drakengren et al. [1997] for the constraint checking parts of the algorithm. The culprit detection part (line 18) does not increase the asymptotic complexity of the algorithm.

Note that the consistency checking may terminate after detecting the first *culprit*, whereas *SAT-TR* runs through all edges in C to find *all* the culprits.

Example 7: Consider the following set of constraint over intervals $\{A, B, C, D, E, F\}$: $A\ (oo^{-1}m^{-1})\ B$, $B\ (o)\ C$, $C\ (o^{-1}f)\ A$, $A\ (\equiv o^{-1})\ D$, $D\ (o^{-1}b^{-1})\ C$, $D\ (\equiv o^{-1}m^{-1})\ E$, $A\ (\equiv)\ E$, $F\ (o\ \ o^{-1}s^{-1}m^{-1})\ E$, $F\ (oo^{-1}d^{-1}s)\ C$, and $F\ (o^{-1}d)\ A$.
The corresponding interval constraint network C is shown in Figure 5 below.

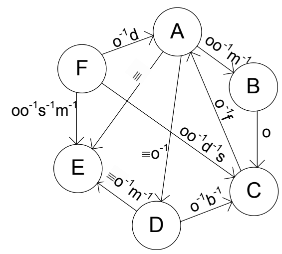


Figure 5: The constraint network C

Algorithm *FindMTSs* will find out which algebras this problem does not belong to. For example, constraint C_1 excludes the following algebras, Table[o][$!b$] ⇒ 9, 12; Table[o][$!s$] ⇒ 3, 5, 7, 16; Table[o][$!b$] ⇒ 9, 12; Table[o][$!d$] ⇒ 13; Table[o][$!f$] ⇒ 6; Table[o][$!eq$] ⇒ 17; Table[o][$!f$] ⇒ 2, 4, 8, 15; Table[o][$!d^{-1}$] ⇒ 10; and Table[o][$!\ s^{-1}$] ⇒ 1.
Algorithm *FindMTSs* will finally return a set for which only the value for Alj[11] remains true. The problem is tractable, and some possible pivots (oo^{-1}) have been identified. We can now use Algorithm *SAT-TR* to decide satisfiability of the problem. First, we must redirect every edge so that C does not have any arc labeled only with o^{-1} pivot (lines 4-5). The new constraint network C is shown in Fig. 6. *C'* is the graph C where every arc labeled with both the pivots are removed (lines 6-7). *C'* is shown in Fig. 7. The next step consists in finding every strong component in C'. These components are shown in Fig. 8.

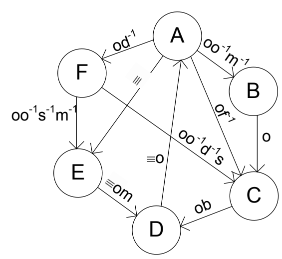


Figure 6: C revised

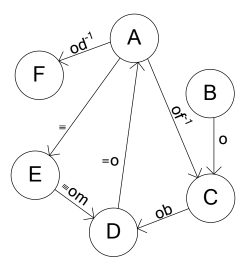


Figure 7: *C'*

Finally, for every arc A in C that does not include the ($\equiv$) relation, return 'reject' if A connects any two nodes in the set of nodes of strong components. This is the case for the arcs labeled *(ob)* and (of^{-1}) in bold on Figure 8b. Therefore, the original problem is not satisfiable and the *culprits* are detected to be $\{(C\ (od)\ D), (A\ (of^{-1})\ C)\}$.

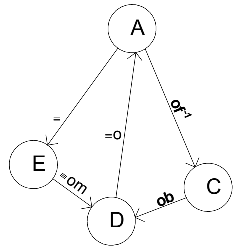


Figure 8: Strong component

The arcs in the *culprit* set are not individually responsible for inconsistency, rather they together cause inconsistency. Since we are detecting all the SCCs leading to an inconsistent TCN, and only the SCCs are responsible for inconsistency (Theorem 3), the *culprit* set is unique.

6. Discussion

In this paper we have completed some gaps left in **Drakengren**, Krokhin, Jeavon, and Johnsson's series of works [Drakengren et al. 1997, and Krokhin et al. 2003] on the Interval Algebra identifying all its MTSs but the OH. We have proposed a mechanism (*FindMTSs*) to classify a network in some or possibly none of the eighteen MTSs. In case it does not belong to any one of them, the problem is intractable [Krokhin et al., 2003]. We have adapted an algorithm from those works to check for consistency when a network does belong to any of those MTSs (except the OH). Even though OH is expected to be the most "useful" algebra because it contains all the atomic relations as its elements, other MTS's may also be useful in practical situations where all the basic relations are not needed (e.g., 1D-atomic relations like '*meet*' are often impractical). Our algorithm not only checks for consistency but in case of inconsistency it returns the arcs (*culprits*) that are responsible for inconsistency. Although Allen's path consistency algorithm (less efficient than Drakengen et al.'s and our algorithms) can check for consistency over tractable problems, it is not easy to modify Allen's algorithm (1983) for identification of culprits or detect intractable problems. In a future work we will develop an algorithm toward the OH MTS that is not covered by the *SAT-TR* algorithm. Algorithm *SAT-TR* is not suitable for OH because there is no fixed pivot in the latter case. A different strategy and more involved algorithm are warranted and we have provided hints on how to address the problem.

An objective of this work is to demonstrate how one may find *culprits* in case of any inconsistent *Qualitative Spatio-temporal Network*, and we hope that this article provides a new direction of research in the latter area, as detection of *culprits* will make a reasoning system more useful.

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Key Terms: Qualitative Temporal Constraint Network; Maximal Tractable Sub-algebra; Consistency algorithm; Culprit detection.

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