Constraint Satisfaction Problems

Chapter 6

Outline

- ♢ CSP examples
- ♢ Backtracking search for CSPs
- ♢ Problem structure and problem decomposition
- ♢ Local search for CSPs

Constraint satisfaction problems (CSPs)

Standard search problem:
state is a "black box"—any old data structure
that supports goal test, eval, successor

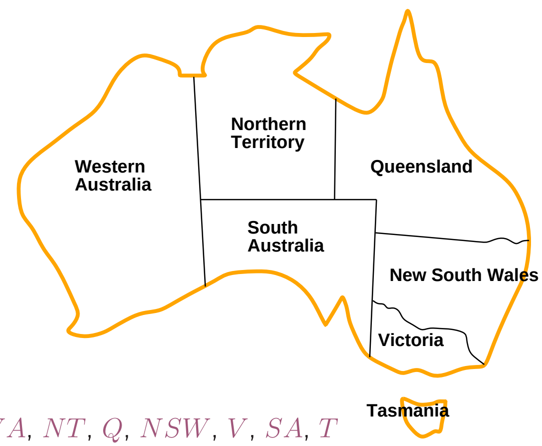
CSP:
state is defined by variables X_i with values from domain D_i

goal test is a set of constraints specifying
allowable combinations of values for subsets of variables

Simple example of a **formal representation language**

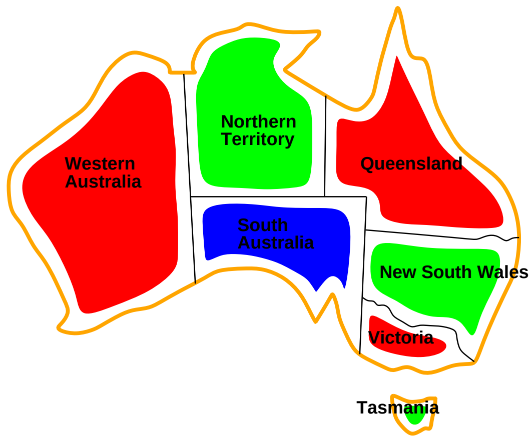
Allows useful **general-purpose** algorithms with more power than standard search algorithms

Example: Map-Coloring



Variables WA, NT, Q, NSW, V, SA, T
Domains $D_i = \{red, green, blue\}$
Constraints: adjacent regions must have different colors
e.g., $WA \neq NT$ (if the language allows this), or
$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

Example: Map-Coloring contd.

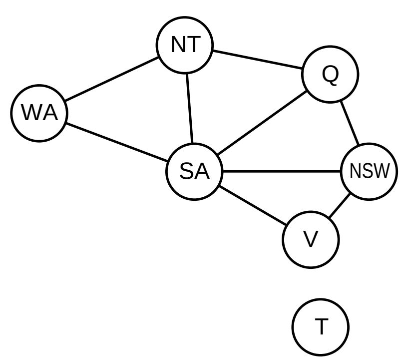


Solutions are assignments satisfying all constraints, e.g.,
$\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint graph

Binary CSP: each constraint relates at most two variables

Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure
to speed up search. E.g., Tasmania is an independent subproblem!

Varieties of CSPs

Discrete variables
- finite domains; size d $\Rightarrow$ $O(d^n)$ complete assignments
  - ♢ e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
- infinite domains (integers, strings, etc.)
  - ♢ e.g., job scheduling, variables are start/end days for each job
  - ♢ need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
  - ♢ linear constraints solvable, nonlinear undecidable

Continuous variables
- ♢ e.g., start/end times for Hubble Telescope observations
- ♢ linear constraints solvable in poly time by LP methods

Varieties of constraints

Unary constraints involve a single variable,
e.g., $SA \neq green$

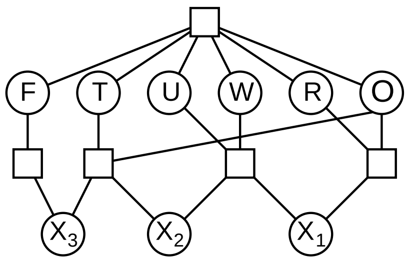
Binary constraints involve pairs of variables,
e.g., $SA \neq WA$

Higher-order constraints involve 3 or more variables,
e.g., cryptarithmetic column constraints

Preferences (soft constraints), e.g., red is better than $green$
often representable by a cost for each variable assignment
→ constrained optimization problems

Example: Cryptarithmetic

```
   T W O
 + T W O
 -------
 F O U R
```



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
Domains: $\{0,1,2,3,4,5,6,7,8,9\}$
Constraints
    $alldiff(F,T,U,W,R,O)$
    $O+O=R+10\cdot X_1$, etc.

Real-world CSPs

Assignment problems
    e.g., who teaches what class, who flies which flight

Timetabling problems
    e.g., which class is offered when and where, which flight is scheduled when and where

Hardware configuration

Spreadsheets

Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables

Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far

◇ Initial state: the empty assignment, $\{\,\}$

◇ Successor function: assign a value to an unassigned variable
    that does not conflict with current assignment.
    ⇒ fail if no legal assignments (not fixable!)

◇ Goal test: the current assignment is complete

1) This is the same for all CSPs!
2) Every solution appears at depth n with n variables (d values each)
    ⇒ use depth-first search
3) Path is irrelevant, so can also use complete-state formulation
4) $b=(n-\ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!!

Backtracking search

Variable assignments are commutative, i.e.,
    $[WA=red$ then $NT=green]$ same as $[NT=green$ then $WA=red]$

Order of the variable assignments is not important, pick an arbitrary order

Consider assignments to a different variable at each level (according to the order)
    ⇒ $b=d$ and there are d^n leaves

Depth-first search for CSPs with single-variable assignments is called backtracking search

Backtracking search is the basic uninformed algorithm for CSPs

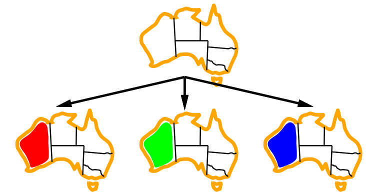
Can solve n-queens for $n\approx 25$

Backtracking search

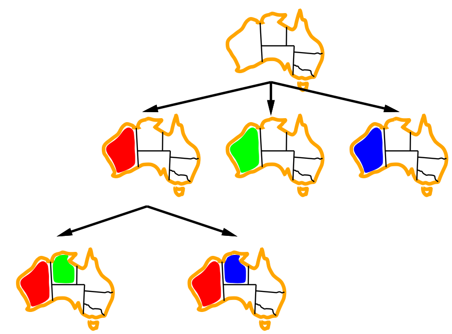
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```


Backtracking example

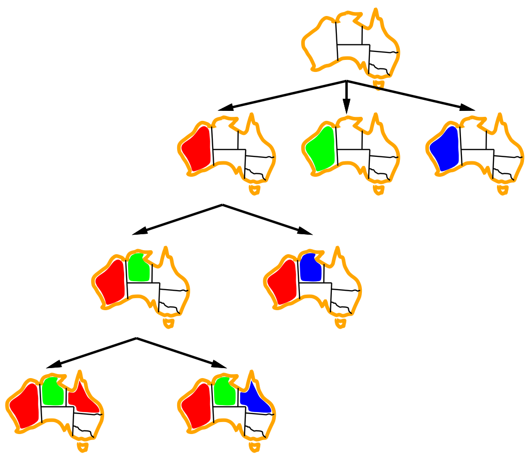
Backtracking example



Backtracking example



Backtracking example



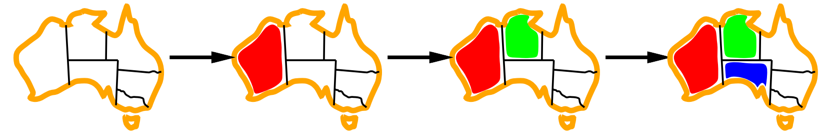
Improving backtracking efficiency

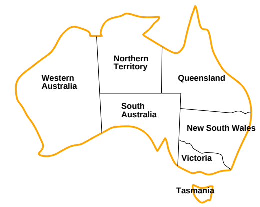
General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Choosing a variable: Minimum remaining values

Minimum remaining values (MRV):
choose the variable with the fewest legal values

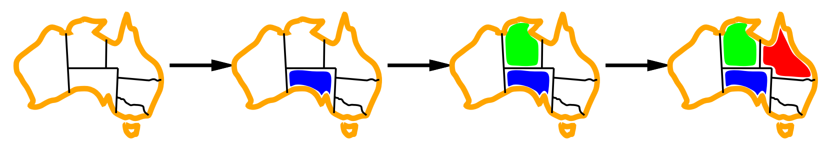




Choosing a variable: Degree heuristic

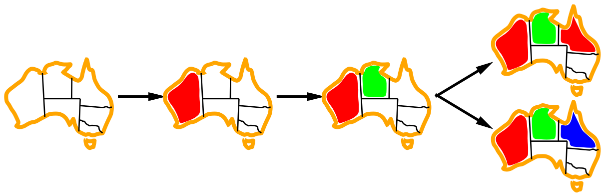
Tie-breaker among MRV variables

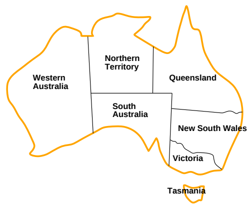
Degree heuristic:
choose the variable with the most constraints on remaining variables (highest degree)



Choosing a value: Least constraining value

Given a variable, choose the least constraining value:
the one that rules out the fewest values in the remaining variables

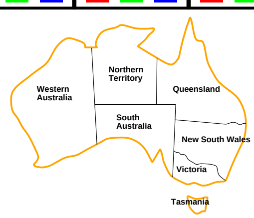




Combining these heuristics (most-constraining variables, least-contraining values) makes 1000 queens feasible

Forward checking (1-step look ahead)

- Keep track of remaining legal values for unassigned variables
  - Help MRV
  - Terminate search when any variable has no legal values



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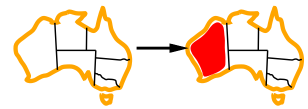

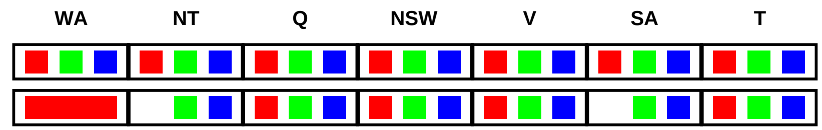

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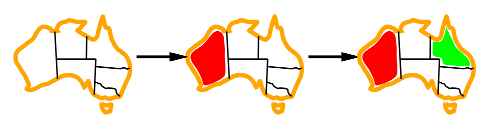

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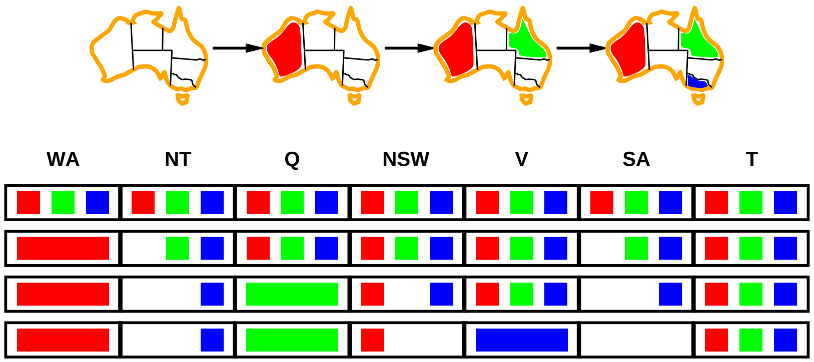
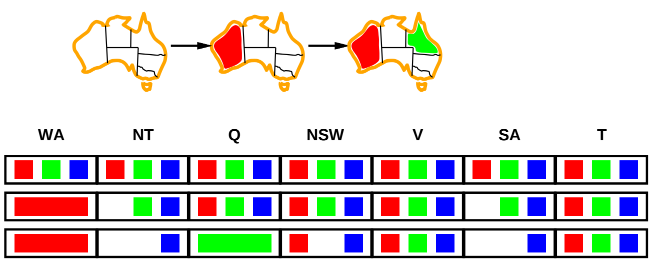


Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



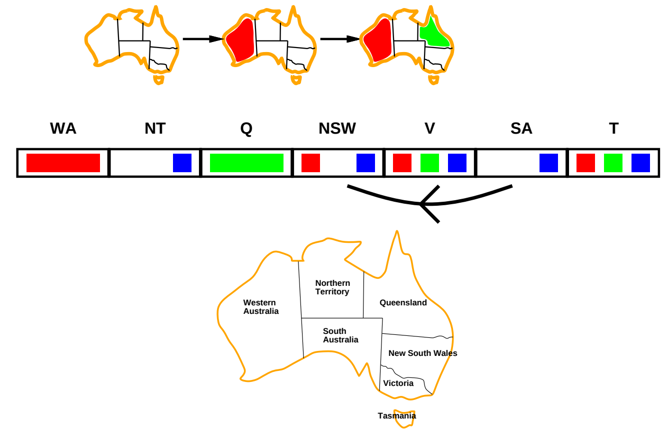
NT and SA cannot both be blue!

Constraint propagation repeatedly enforces constraints locally

Arc consistency (multi-step look ahead)

Simplest form of propagation makes each arc consistent

$X \rightarrow Y$ is consistent iff
for **every** value x of X there is **some** allowed y



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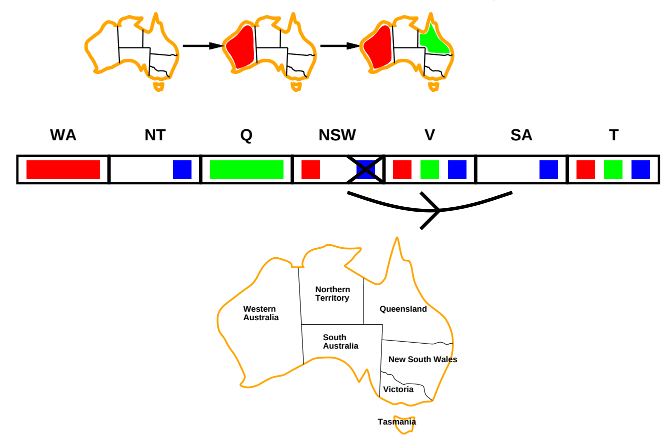

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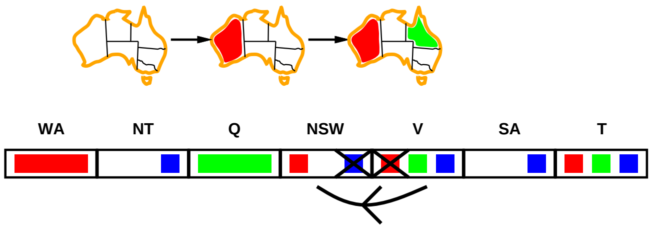

If X loses a value, neighbors of X need to be rechecked

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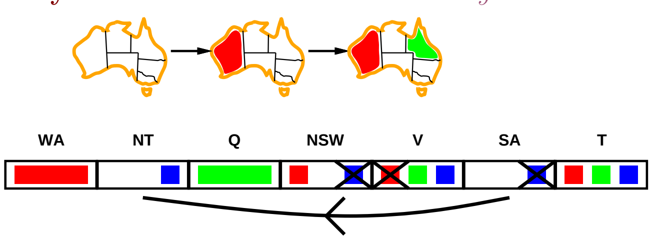


If X loses a value, neighbors of X need to be rechecked

Arc consistency detects failure earlier than forward checking

Can be run as a preprocessor or after each assignment

Arc consistency algorithm

function AC-3(csp) **returns** the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: $queue$, a queue of arcs, initially all the arcs in csp

  while $queue$ is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST($queue$)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to $queue$

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  $removed \leftarrow false$
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN[X_i]; $removed \leftarrow true$
  return $removed$

$O(n^2 d^3)$: n^2 arcs, d enqueue's, d^2 pairs of values to check [skip p.147-9]

Iterative algorithms for CSPs

Hill-climbing, simulated annealing typically work with
"complete" states, i.e., all variables assigned

To apply to CSPs:
  allow states with unsatisfied constraints
  operators **reassign** variable values

Variable selection: randomly select any conflicted variable

Value selection by min-conflicts heuristic:
  choose value that violates the fewest constraints
  i.e., hillclimb with $h(n)$ = total number of violated constraints

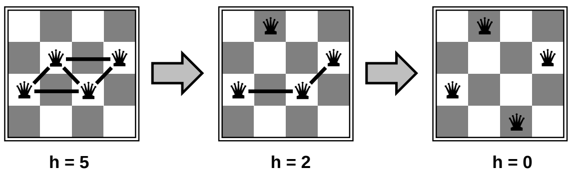
Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)

Operators: move queen in column

Goal test: no attacks

Evaluation: $h(n)$ = number of attacks

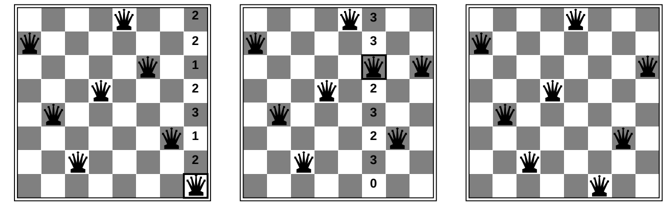


MIN-CONFLICTS Algorithm

function MIN-CONFLICTS(*csp*, *max-steps*) **returns** a solution or failure
  inputs: *csp*, a constraint satisfaction problem
    max-steps, the number of steps allowed before giving up
  local variables: *current*, a complete assignment
    var, a variable
    value, a value for a variable

  current ← an initial complete assignment for *csp*
  for $i = 1$ to *max-steps* **do**
    var ← a randomly chosen, conflicted variable from VARIABLES[*csp*]
    value ← the value *v* for *var* that minimizes CONFLICTS(*var*, *v*, *current*, *csp*)
    set *var*=*value* in *current*
    if *current* is a solution for *csp* **then return** *current*
  end
  return *failure*

Example: 8-Queens Min conflicts

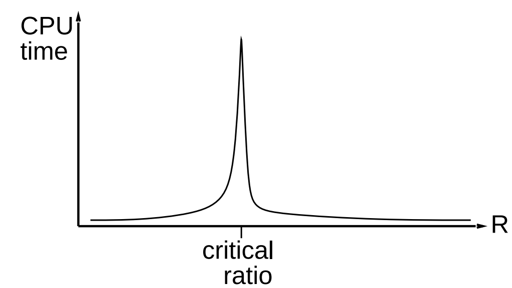


Performance of min-conflicts

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

The same appears to be true for any randomly-generated CSP **except** in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

CSPs are a special kind of problem:
states defined by values of a fixed set of variables
goal test defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies

Iterative min-conflicts is usually effective in practice