Reducibility

There are two main techniques for showing that problems are undecidable:

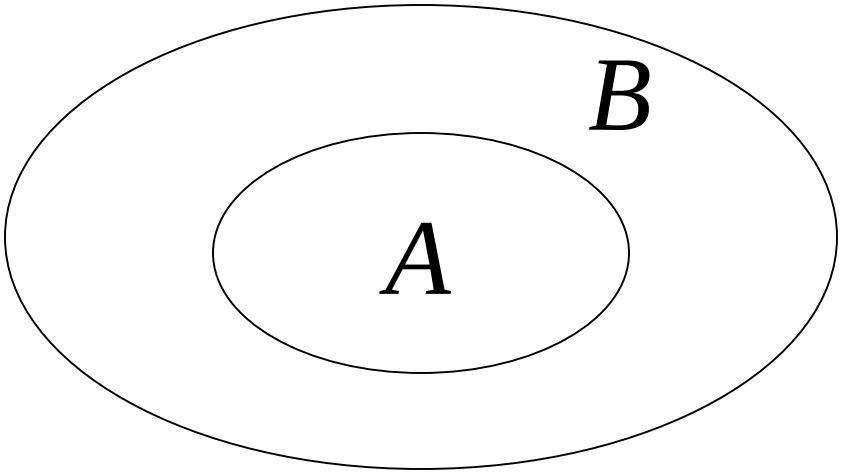
diagonalization and *reduction*

We say that a problem *A* is **reduced** to a problem B if the decidability of *A* follows from the decidability of B.

Linz, 6th, page 315

Problem A is reduced to problem B

If we can solve problem B then we can solve problem A



A is reducible to B if we can use B as a subroutine to solve A.

Use a halting TM for B in the construction of a halting TM which decides problem A.

Problem A is reduced to problem B

$\Downarrow \Uparrow$

If B is decidable then A is decidable

$\Downarrow \Uparrow$

If A is undecidable then B is undecidable

Example: the halting problem

is reduced to

the state-entry problem

So, if the halting problem is undecidable, then the state-entry problem is undecidable.

The state-entry problem

Inputs:

- Turing Machine M
- State q
- String w

Question: Does M enter state q on input w ?

Theorem:

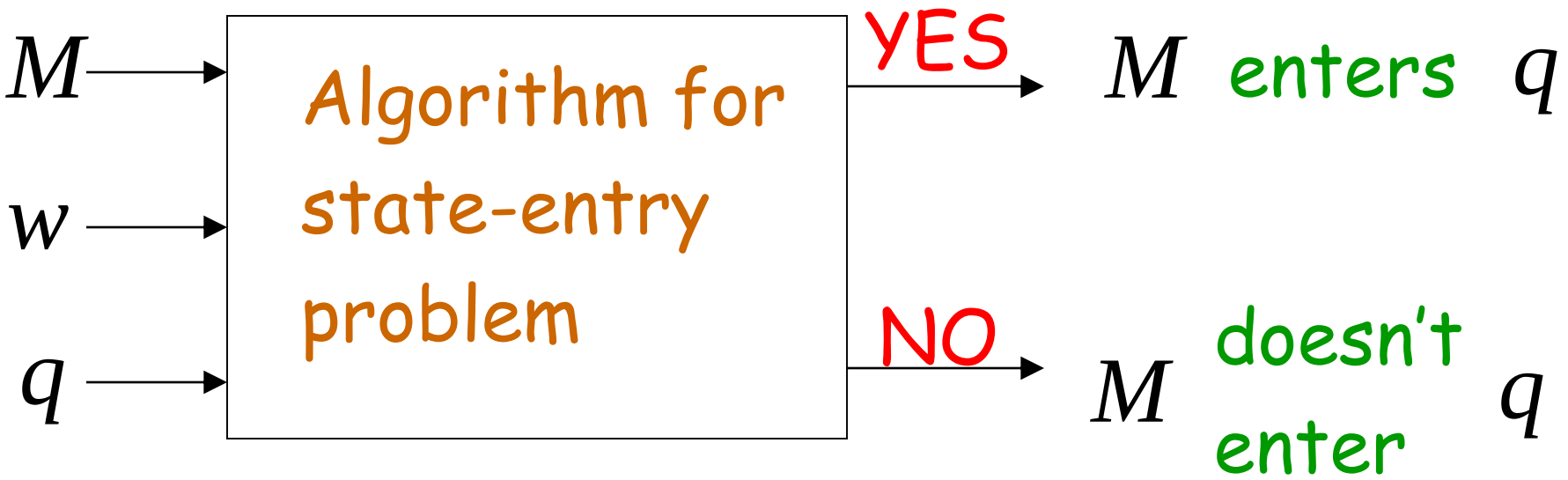
The state-entry problem is undecidable

Proof: Reduce the halting problem to the state-entry problem.

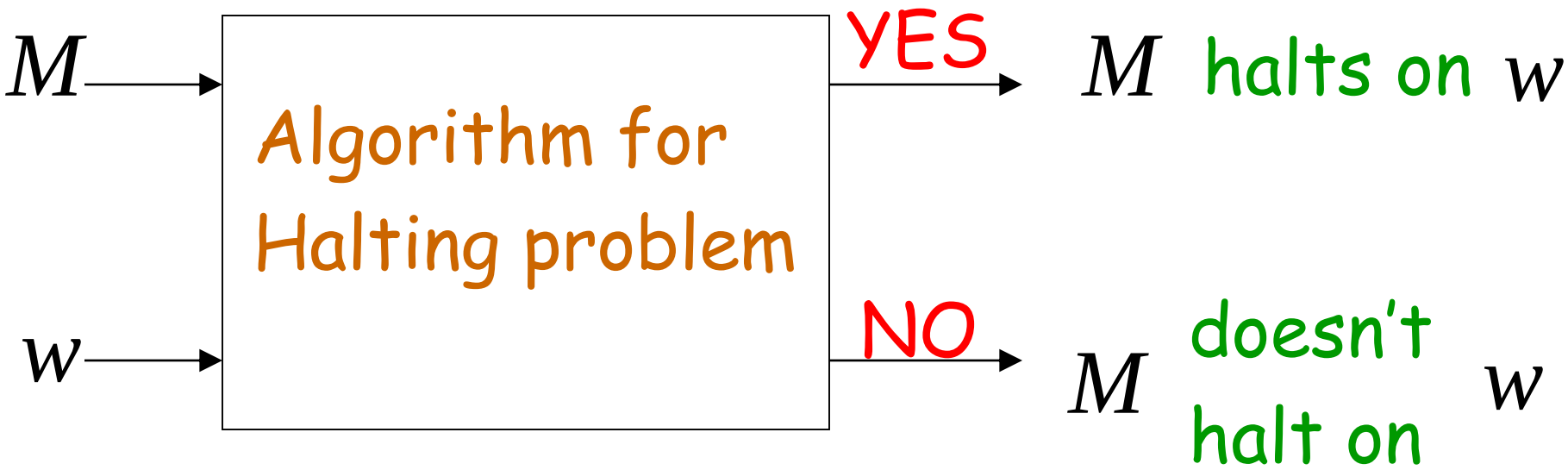
Suppose we have an algorithm (Turing Machine) that solves the state-entry problem

We will construct an algorithm that solves the halting problem

Assume we have the state-entry algorithm:

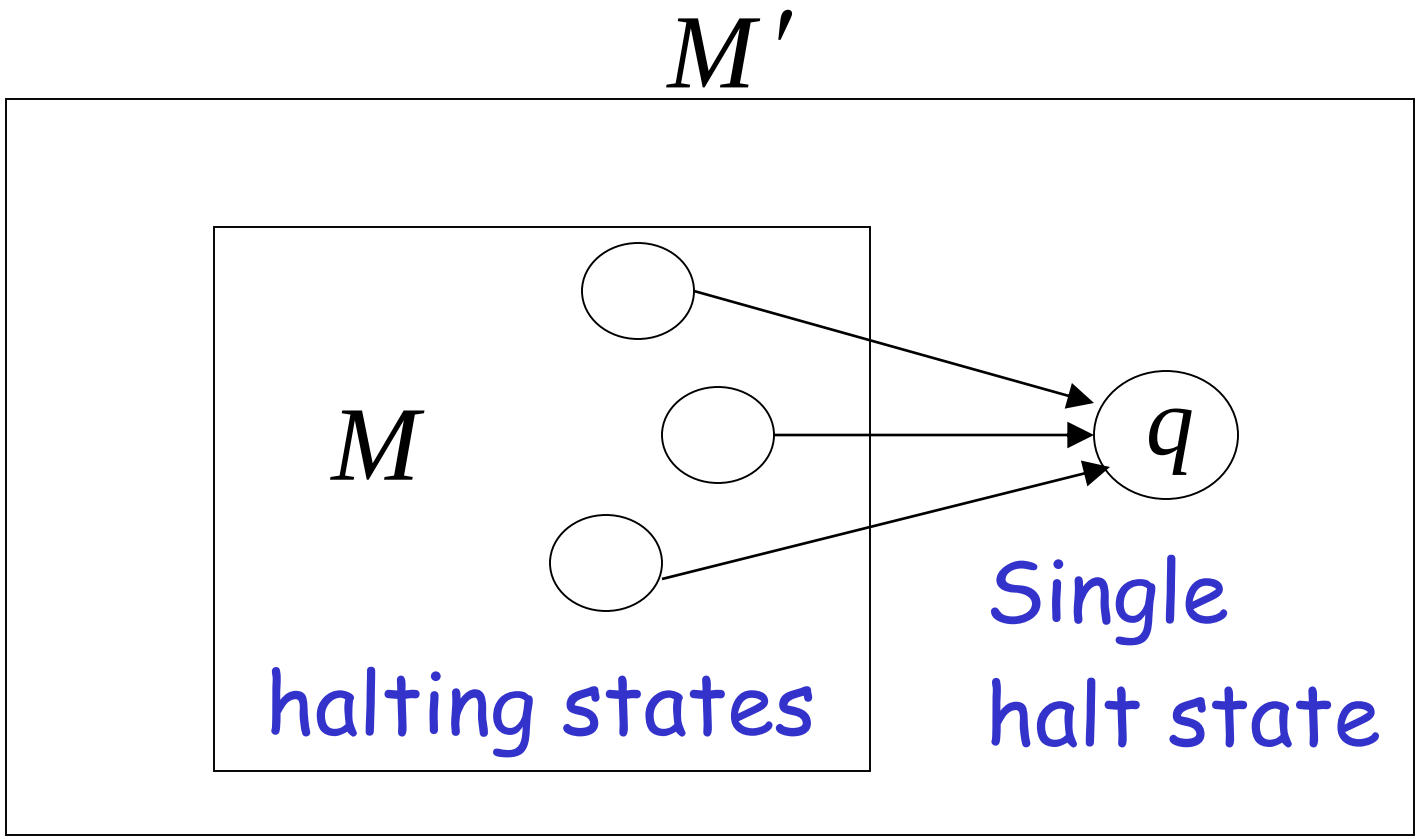


We want to design the halting algorithm:



Modify any machine M to construct M':

- Add new state q
- From any halting state add transitions to q



M halts

if and only if

M' halts on state q

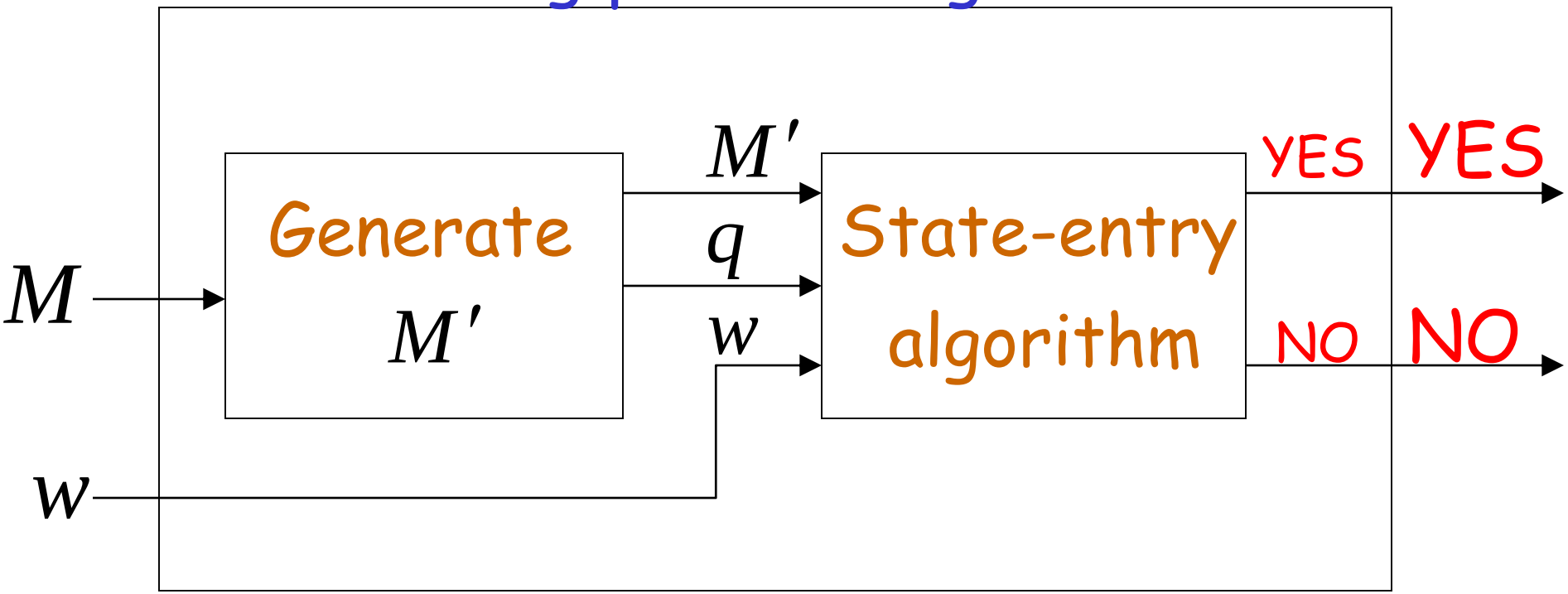
Algorithm for halting problem:

Inputs: machine M and string w

1. Construct machine M' with state q

2. Run algorithm for state-entry problem with inputs: M' , q , w

Halting problem algorithm



We reduced the halting problem to the state-entry problem

Since the halting problem is undecidable, it must be that the state-entry problem is also undecidable

END OF PROOF

Another example:

the halting problem

is reduced to

the blank-tape halting problem

The blank-tape halting problem

Input: Turing Machine M

Question: Does M halt when started with a blank tape?

Theorem:

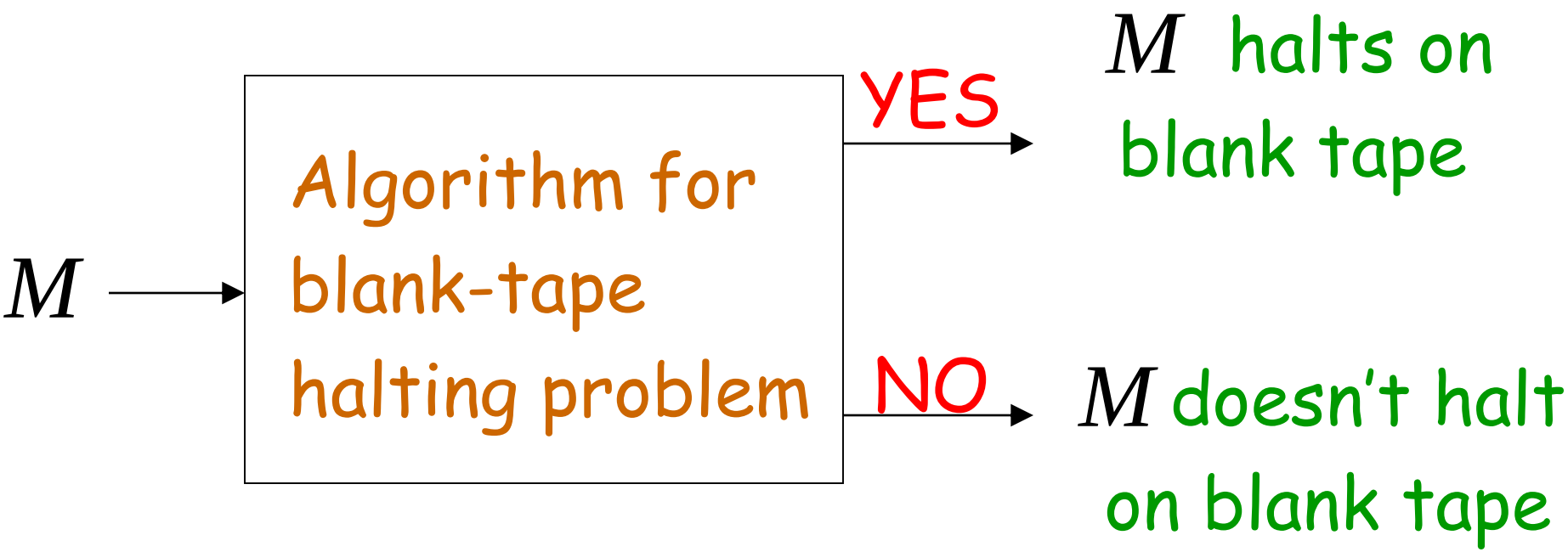
The blank-tape halting problem is undecidable

Proof: Reduce the halting problem to the blank-tape halting problem

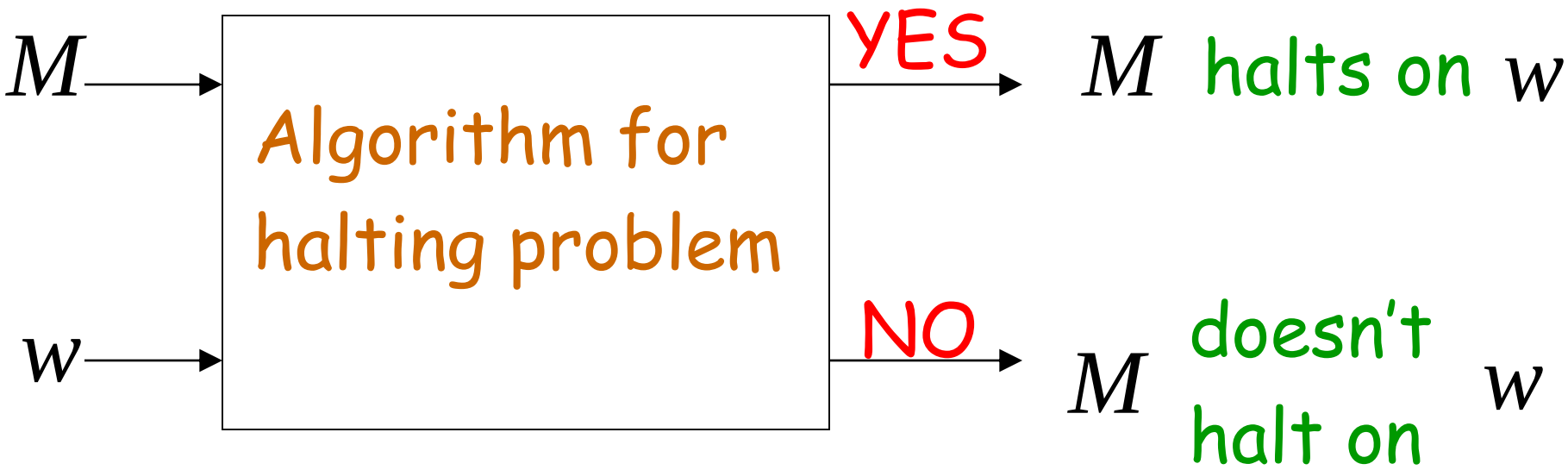
Suppose we have an algorithm
for the blank-tape halting problem

We will construct an algorithm
for the halting problem

Assume we have the blank-tape halting algorithm/machine. It looks like this:



We want to design a machine that decides the halting problem. The machine looks like this:



Construct a new machine M_w

- On blank tape writes w
- Then continues execution like M

M_w

step 1	step2
if blank tape then write w	execute M with input w

M halts on input string w

if and only if

M_w halts when started with blank tape

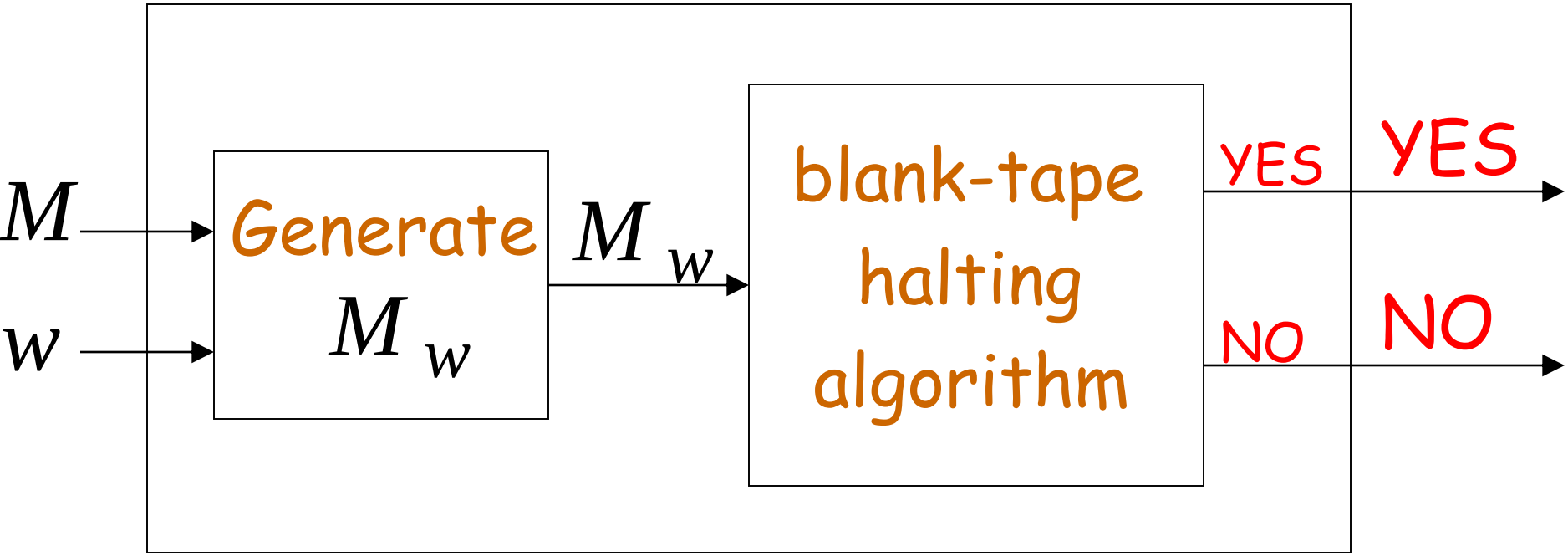
Algorithm for halting problem:

Inputs: machine M and string w

1. Construct M_w

2. Run algorithm for blank-tape halting problem with input M_w

Halting problem algorithm



We reduced the halting problem
to the blank-tape halting problem

Since the halting problem is undecidable,
the blank-tape halting problem is
also undecidable

END OF PROOF

Summary of Undecidable Problems

Halting Problem:

Does machine M halt on input w ?

Membership problem:

Does machine M accept string w ?

In other words: Is a string w member of a recursively enumerable language L ?

Blank-tape halting problem:

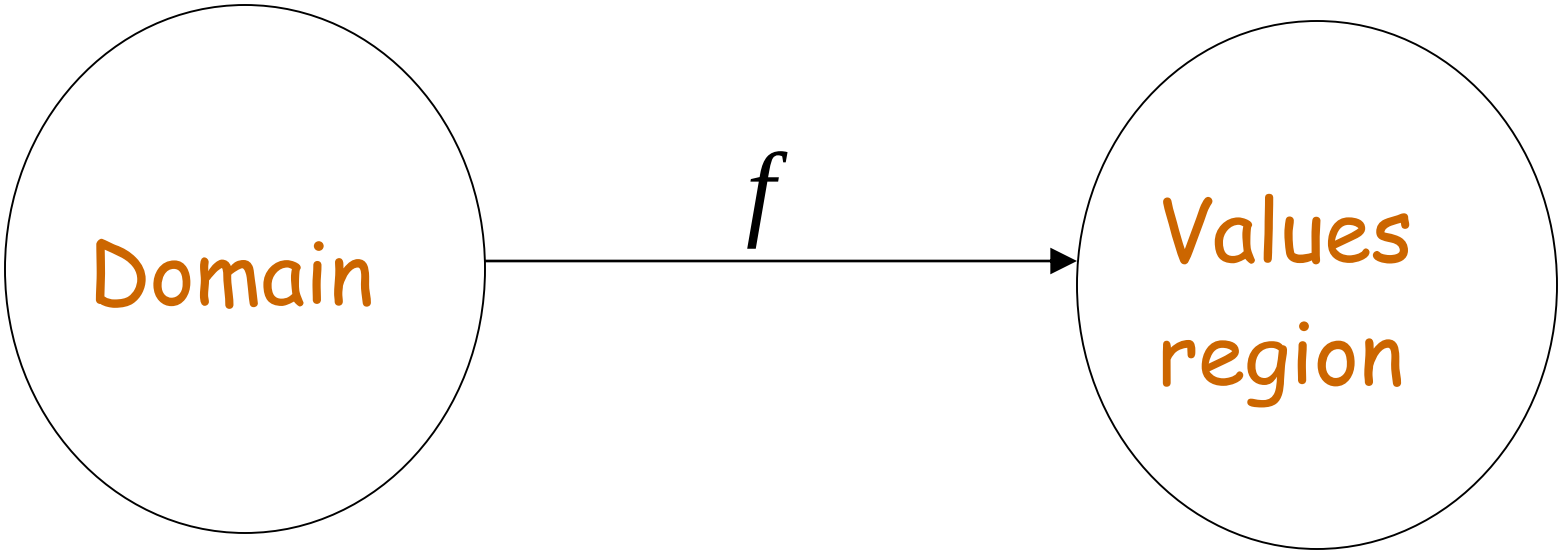
Does machine M halt when starting on blank tape?

State-entry Problem:

Does machine M enter state q on input w ?

Uncomputable Functions

Uncomputable Functions



A function is uncomputable if it cannot be computed for all of its domain

An uncomputable function:

$$f(n) = \begin{cases} \text{maximum number of moves until} \\ \text{any Turing machine with } n \text{ states} \\ \text{halts when started with the blank tape} \end{cases}$$

Theorem: Function $f(n)$ is uncomputable

Proof: Assume for contradiction that $f(n)$ is computable

Then the blank-tape halting problem is decidable

Algorithm for blank-tape halting problem:

Input: machine M

1. Count states of M : m

2. Compute $f(m)$

3. Simulate M for $f(m)$ steps starting with empty tape

If M halts then return YES
otherwise return NO

Therefore, the blank-tape halting problem is decidable

However, the blank-tape halting problem is undecidable

Contradiction!!!

Therefore, function $f(n)$ in uncomputable

END OF PROOF

Rice's Theorem

Definition:

Non-trivial properties of recursively enumerable languages:

any property possessed by some (not all) recursively enumerable languages

Some non-trivial properties of recursively enumerable languages:

- L is empty
- L is finite
- L contains two different strings of the same length

Rice's Theorem:

Any non-trivial property of
a recursively enumerable language
is undecidable

Rice's Theorem

In exactly the same manner, we can substitute other questions such as "Does *L(M)* contain any string of length five?" or "Is *L(M)* regular?" without affecting the argument essentially. These questions, as well as similar questions, are all undecidable. A general result fromalizng this is known as **Rice's theorem**.

Rice's Theorem

This theorem states that any nontrivial property of a recursively enumerable language is undecidable. The adject "nontrivial" refers to a property possed by some but not all recursively enumerable languages. A precide statement and a proof of Rice's theory can be found in Hopcroft and Ullman (1979).

Linz, 6th, pages 321-322

A property of the RE languages is simply a set of RE languages. Thus, the property of being context-free is formally the set of all CFL's. The property of being empty is the set $\{\emptyset\}$ consisting of only the empty language.

Theorem 9.11: (Rice's Theorem) Every nontrivial property of the RE languages is undecidable.

Hopcroft, Motwani, Ullman, 3rd, pages 397-398

We will prove some non-trivial properties without using Rice's theorem

Theorem:

For any recursively enumerable language L it is undecidable to determine whether L is empty

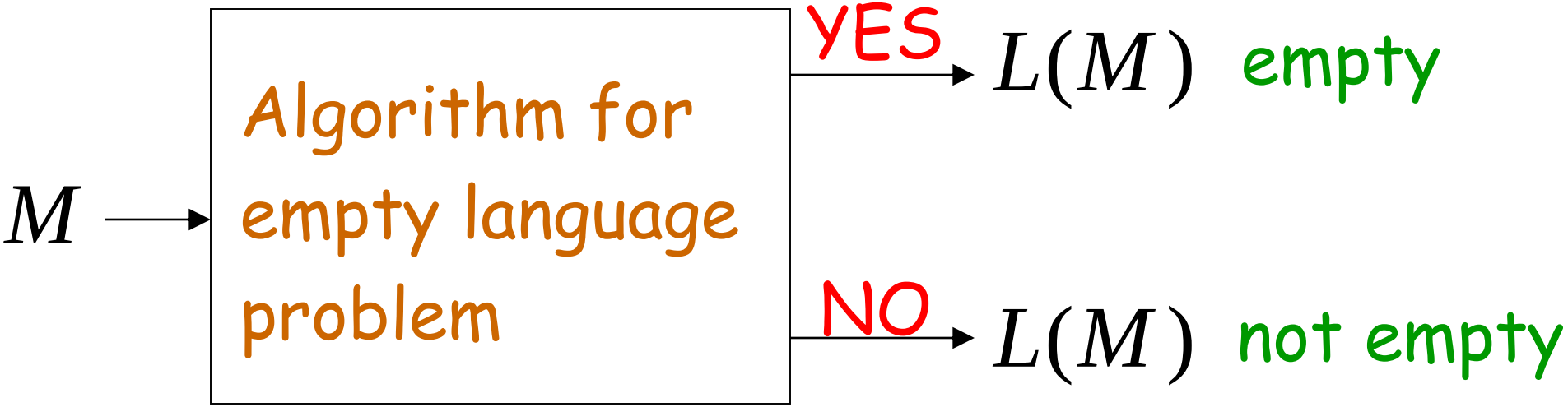
Proof:

We will reduce the membership problem to this problem

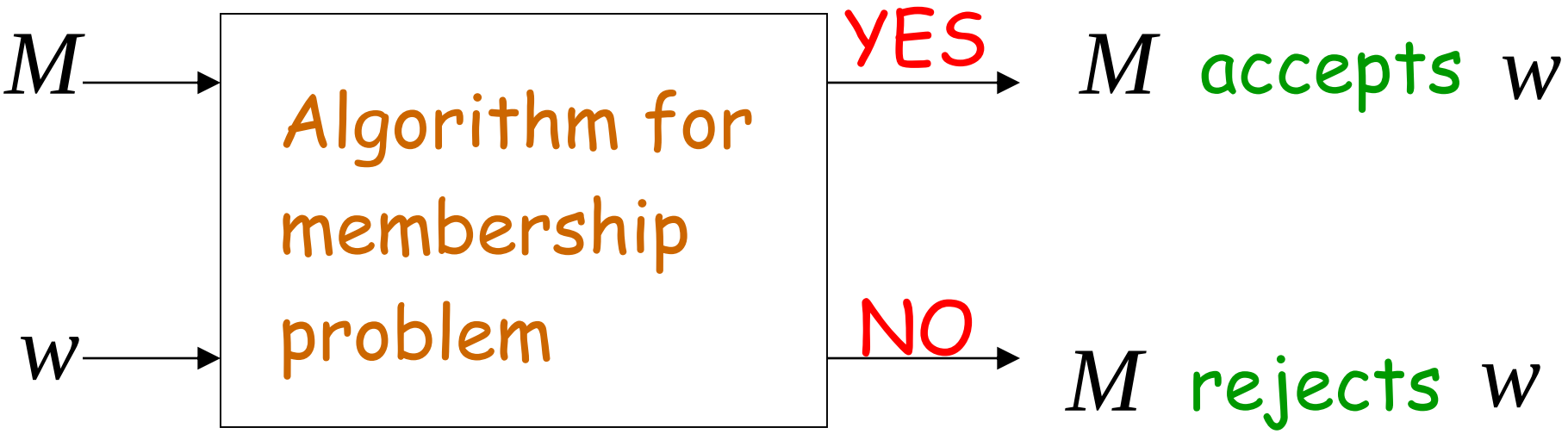
Let M be the machine that accepts L

$$L(M) = L$$

Assume we have the empty language algorithm:



We will design the membership algorithm:



First construct machine M_w:

When M enters a final state,
compare original input string with w

Accept if original input is
the same with w

$$w \in L$$

if and only if

$L(M_w)$ is not empty

$$L(M_w) = \{w\}$$

Algorithm for membership problem:

Inputs: machine M and string w

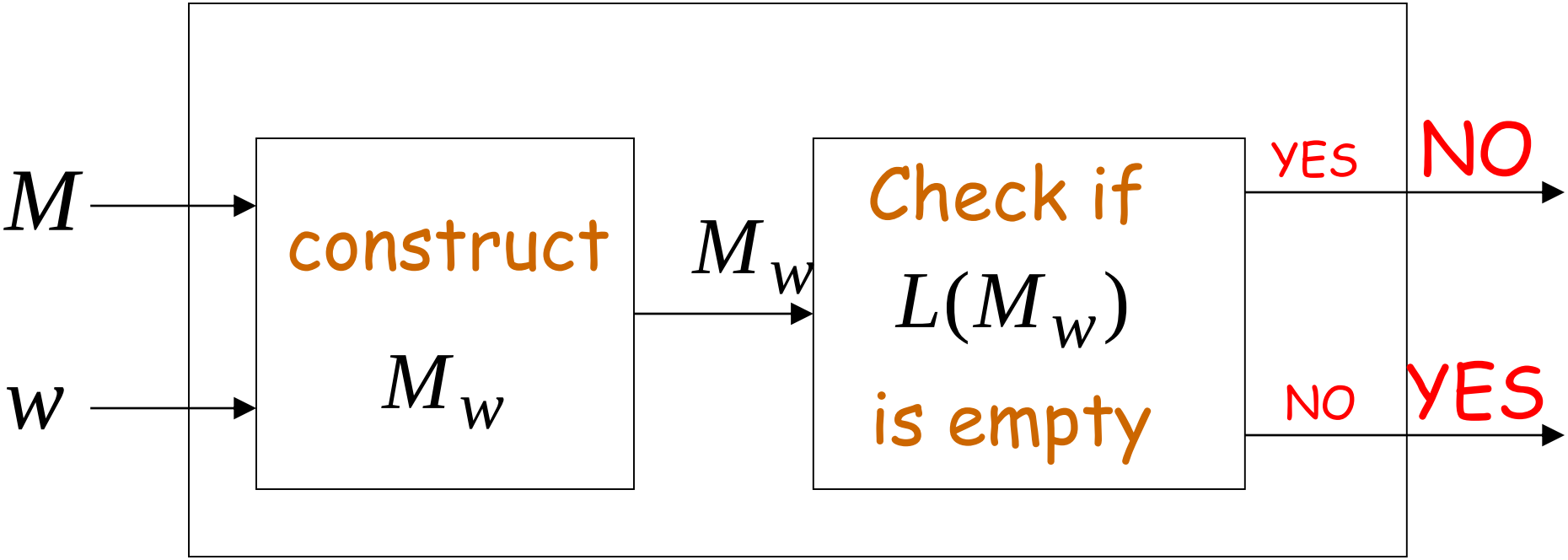
1. Construct M_w

2. Determine if $L(M_w)$ is empty

YES: then $w \notin L(M)$

NO: then $w \in L(M)$

Membership algorithm



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