CSE 1400 Applied Discrete Mathematics
Mathematical Induction

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Abstract

The word induction has many meanings. For us it is a formal proof process that a predicate $\mathrm{P}(n)$ is `True` for all natural numbers n belonging to some set, most often the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The principle of induction is: If set $\mathbb{X}$ contains zero and if $x \in \mathbb{X}$ implies its successor $x + 1 \in \mathbb{X}$, then every natural number is in $\mathbb{X}$, that is $\mathbb{X} = \mathbb{N}$.

Induction

Induction often involves sums:

> If the sum of n term equals some formula, then the sum of $n + 1$ terms equals the same formula with appropriate substitutions.

Induction often involves recurrsion:

> A function $f(n)$ satisfies a recurrence relation.

An inductive proof of the truth of predicate $\mathrm{P}(n)$ for all natural numbers n has two steps.

1. A proof that $\mathrm{P}(0)$ is `True`.
2. A proof of the truth of the conditional statement

   If $\mathrm{P}(n)$ is `True`,

   then $\mathrm{P}(n+1)$ is `True`.

A short notation for this is

$$\begin{array}{ll} & \mathrm{P}(0) \\ & \mathrm{P}(n) \to \mathrm{P}(n+1) \\ \hline \therefore & (\forall n \in \mathbb{N})\mathrm{P}(n) \end{array}$$

Which is read "$\mathrm{P}(0)$ is `True`; and if $\mathrm{P}(n)$ is `True`, then $\mathrm{P}(n+1)$ is `True`. Therefore, $\mathrm{P}(n)$ is `True` for all natural numbers n."

Summations

Consider the predicate:

> The sum of the first n natural numbers is $n(n-1)/2$.

For instance, the sum of the first 5 natural numbers is

$$0+1+2+3+4 = 10 = 5 \cdot (5-1)/2 = 5 \cdot 4/2$$

Since the sum of the first 5 natural numbers is 10, the sum of the first 6 natural numbers is

$$\begin{aligned} 10+5 &= 15 \\ &= 6 \times \frac{5}{2} \end{aligned}$$

You could use the notation $\text{P}(n)$ to denote the truth value of the statement

"The sum of the first n natural numbers is $n(n-1)/2$."

The heart of mathematical induction relies on (1) noticing a pattern for small initial cases, and (2) establishing a link allowing a proof of of an instance from the truth of previous instances.

In this case, the pattern is

$$0+1+2+3+4+\cdots+(n-1) = n \times \frac{n-1}{2}$$

The sum of n terms in an arithmetic progression is n times the average of the first and last term.

which can be phrases as "the sum of the first n natural numbers is n times $n-1$ divided by 2." or $\text{P}(n)$ for short.

To establish the link between the truth of $\text{P}(n)$ and $\text{P}(n+1)$ requires a little algebra. We must generalize the instance above where the step was from 5 to 6. If the sum of the first n natural numbers is $n(n-1)/2$, then the sum of the first $n+1$ natural numbers is $n(n-1)/2$ plus n. That is, if

The sum of the first $n+1$ natural numbers is the sum of the first n natural numbers plus n, the $(n+1)^{\text{st}}$ natural number.

$$0+1+2+3+4+\cdots+(n-1) = \frac{n(n-1)}{2}$$

then

$$\begin{aligned} [0+1+2+\cdots+(n-1)]+n &= \frac{n(n-1)}{2}+n \\ &= n\left[\frac{n-1}{2}+1\right] \\ &= (n+1)\frac{n}{2} \end{aligned}$$

This shows that if the function $n(n-1)/2$ correctly computes the value in the nth case, then it correctly computes the value in the $(n+1)^{\text{st}}$ case.

This fact about the sum of natural numbers has a classic "proof by pictures."

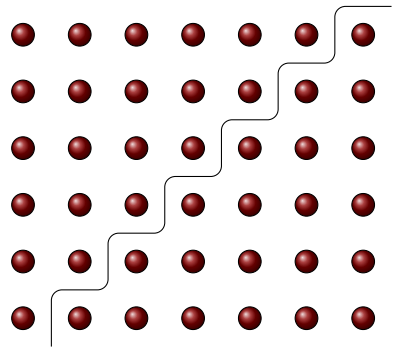
The statement: "The sum of the first n natural numbers is $n(n-1)/2$" is a predicate: A proposition whose truth value depends on one or more variables, in this case, one variable named n. Many

Another example of a predicate is

x squared minus 1 equals 0.

This predicate is `True` for $x = \pm 1$ and `False` for all other values of x.

$$0+1+2+3+4+5+6 = 21 = \frac{1}{2}7(7-1)$$



$$\sum_{k=0}^{n-1} k = 0+1+2+\cdots+(n-1) = \frac{1}{2}n(n-1)$$

Table 1: The number of dots in the rectangle is $n \times (n-1)$. One-half of the dots sums the natural numbers from 1 to $(n-1)$.

predicates, especially predicates about summations and recurrence equations, can be proved `True` by mathematical induction.

SUMMATION NOTATION IS USED TO WRITE PHRASES SUCH AS "THE SUM OF THE FIRST n NATURAL NUMBERS." The summation notation for this phrase is

$$\sum_{0 \le k < n} k$$

The phrase "the sum of the first n powers of 2" is written

$$\sum_{0 \le k < n} 2^k$$

The phrase "the sum of the first n odd natural numbers" is written

$$\sum_{0 \le k < n} (2k+1)$$

In each of these examples, k is a "dummy variable" that successively takes on the values 0, 1, 2, up to and including $(n-1)$. The $\sum$ symbol stands for "sum the values" of the terms which are k, 2^k and $(2k+1)$ in the three examples. Thus,

$$\sum_{0 \le k < n} k = 0+1+2+3+4+\cdots+(n-1)$$

$$\sum_{0 \le k < n} 2^k = 2^0+2^1+2^2+2^3+2^4+\cdots+2^{(n-1)}$$

$$\sum_{0 \le k < n} (2k+1) = 1+3+5+7+9+\cdots+(2n-1)$$

When $n = 0$, the dummy variable k ranges over absurd values 0 up to and including -1. By convention, we take this sum to mean the sum of no values and interpret the sum's value to be 0. That is,

$$\sum_{0 \le k < 0} t_k \equiv 0$$

This sum is called the empty sum and its values is equal to 0 no matter the values of the terms t_k.

Let's give proofs by mathematical induction of the following predicates.

1. "The sum of the first n powers of 2 is $2^n - 1$."

$$\sum_{0 \le k < n} 2^k = 2^n - 1 \tag{1}$$

Basis for Induction: The sum of the first $n = 0$ powers of 2 is the empty sum

$$\sum_{0 \le k < 0} 2^k$$

and equal to 0. The right-hand side of equation 1 is the value of the function $2^n - 1$ at $n = 0$, that is, $2^0 - 1$, which is also equal to 0.

As additional confirmation of the predicate in initial cases note

(a) The sum of the first $n = 1$ powers of 2 is $2^0 = 2^1 - 1$

(b) The sum of the first $n = 2$ powers of 2 is $2^0 + 2^1 = 3 = 2^2 - 1$.

Inductive Premise: Pretend that for some unspecified n, the sum of the first n powers of 2 is $2^n - 1$.

We know that the sum of the first n powers of 2 is $2^n - 1$ is true for $n = 0$, 1 and 2.

Inductive Conclusion: If the sum of the first n powers of 2 is $2^n - 1$, then the sum of the first $(n+1)$ powers of 2 is $2^n - 1$ plus 2^n. That is,

$$\text{If} \quad \sum_{0 \le k < n} 2^k = 2^n - 1$$

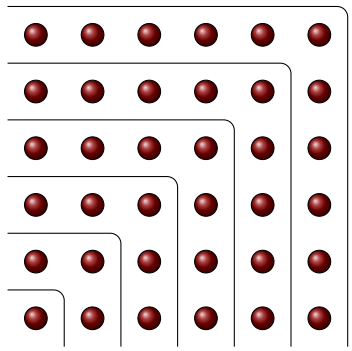
then

$$\begin{aligned}\sum_{0 \le k < (n+1)} 2^k &= \sum_{0 \le k < n} 2^k + 2^n \\ &= (2^n - 1) + 2^n \\ &= 2 \cdot 2^n - 1 \\ &= 2^{n+1} - 1\end{aligned}$$

That is, if the function $2^x - 1$ correctly computes the value of the nth case, then it correctly computes the value of the $(n+1)^{\text{st}}$ case.

2. "The sum of the first n odd natural numbers $(n+1)^2$." This fact about the sum of odd natural numbers has a classic "proof by pictures," see Table 2.

$$1+3+5+7+9+11 = 36 = 6^2$$



$$\sum_{k=0}^{n-1}(2k+1) = 1+3+5\cdots+(2n-1) = n^2$$

Table 2: The number of dots in successive squares is the sum of successive odd natural numbers from 1 to $(2n-1)$.

Know that an implicit use of induction is occurring when $\cdots$ are used.

We can reduce the problem of summing odd numbers to problem of summing natural numbers. That is, the sum of the first n odd natural numbers is

$$\begin{aligned}
1+3+5+7+\cdots+(2(n-1)+1) =& (2\cdot 0+1)+(2\cdot 1+1)+(2\cdot 2+1)+(2\cdot 3+1)+\cdots+(2(n-1)+1)\\
=& 2(0+1+2+3+\cdots+(n-1))+(1+1+1+1+1+\cdots+1)\\
=& 2(n(n-1)/2)+n\\
=& n^2
\end{aligned}$$

Here is an inductive proof that

$$\sum_{k=0}^{n-1}(2k+1) = n^2. \tag{2}$$

Basis for Induction: The sum of the first $n = 0$ odd natural numbers is the empty sum

$$\sum_{0\le k<0}(2k+1)$$

and equal to 0. The right-hand side of the equation is the value of the function n^2 at $n = 0$, that is, the left-hand side and the right-hand side of equation 2 are both equal to 0 when $n = 0$. As additional confirmation of the predicate in initial cases note

(a) The sum of the first $n = 1$ odd natural numbers is

$$1 = 1^2.$$

(b) The sum of the first $n = 2$ odd natural numbers is

$$1 + 3 = 2^2.$$

(c) The sum of the first $n = 3$ odd natural numbers is

$$1 + 3 + 5 = 3^2.$$

Inductive Premise: Pretend that for some unspecified n, the sum of the first n odd natural numbers is n^2.

Inductive Conclusion: If the sum of the first n odd natural numbers is n^2, then the sum of the first $(n+1)$ odd natural numbers is n^2 plus $(2n+1)$. That is,

$$\text{If} \quad \sum_{0 \le k < n} (2k+1) = n^2$$

then

$$\begin{aligned} \sum_{0 \le k < (n+1)} (2k+1) &= \sum_{0 \le k < n} (2k+1) + (2n+1) \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

That is, if the function x^2 correctly computes the value in the nth case, then it correctly computes the value in the $(n+1)^{\text{st}}$ case.

3. "There are 2^n bit string of length n."

Basis for Induction: There is $1 = 2^0$ bit strings of length n. It is the empty string, often named λ.

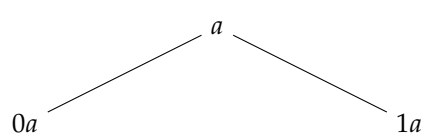
As additional confirmation of the predicate in initial cases note

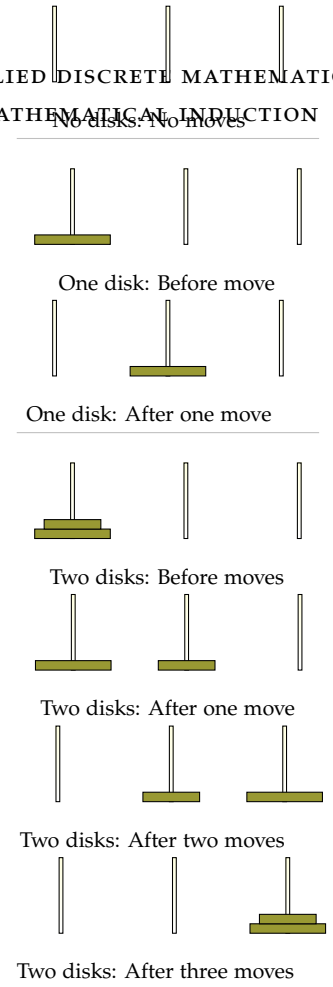
(a) There are 2 bit strings of length 1: 0 and 1.

(b) There are $4 = 2^2$ bit strings of length 2: 00, 01, 10, and 11.

Inductive Premise: Pretend that for some unspecified n, there are 2^n bit strings of length n.

Inductive Conclusion: Consider each bit string a of length n. The string a can be made into a bit string of length $n+1$ by appending a 0 and it can be made into another bit string of length $n+1$ by appending a 1.





That is, every bit string of length n gives rise to 2 different bits strings of length $n+1$. Therefore, if there are 2^n bit strings of length n, there are $2 \cdot 2^n = 2^{n+1}$ bit strings of length $n+1$.

The Tower of Hanoi and Mersenne Numbers

Some problems are just fanciful, such as the Tower of Hanoi. Somewhere in southeast Asia, near Hanoi, monks are moving 64 golden disks from one diamond needle onto another subject to God's laws.

1. Move only one disk at a time.
2. Never place a large disk on a smaller one.

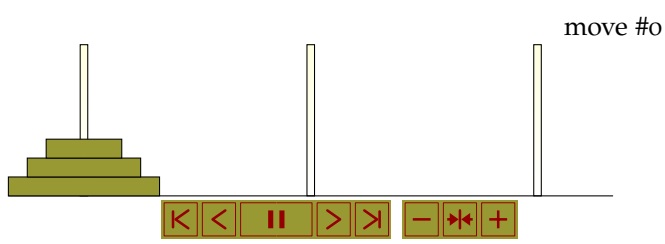
When the monks complete their task, the world will end.

If the monks can move 1 disk every day, we can compute the end of the world by computing the total number of disks they must move. As God instructed, it is helpful to name things, so let n be the number of disks, and let m_n be the number of moves the monks will need to make. For small values of n, the number of moves can be computed by examining them all.

Number of Disks	$n =$	0	1	2	3	4	5	6
Number of Moves	$m_n =$	0	1	3	7	15	31	63

The trick to discovering the recurrence equation that MODELS this problem is to notice that to move the bottom disk the disks on top of it must be moved twice: once off of the bottom disk and once back on to the bottom disk. Convince yourself that for 4 disks the top 3 disks must be moved twice (off-of and on-to the bottom disk) and the bottom disk must be moved only once. In notation,

$$m_4 = m_3 + 1 + m_3 = 2m_3 + 1$$



In general, the number of moves m_n can be computed from the recurrence equation

$$m_n = 2m_{n-1} + 1$$

with initial condition $m_0 = 0$.

If you've studied the example, perhaps you've noticed the Mersenne numbers are computed by the function $m_n = 2^n - 1$. Mathematical induction furnishes a proof the function solves the recurrence equation.

The numbers $m_n = 2^n - 1$, $n = 0, 1, 2, \ldots$ are called Mersenne numbers. They are fundamental in computing.

Oh yes, moving one disk per day, the world will end in $2^{64} - 1$ days $\approx 10^{19.2}$ days $\approx 4.3 \times 10^{16}$ years, that's about 43 quadrillion years.

Basis: For $n = 0$, there are no moves and $m_0 = 2^0 - 1 = 0$ also.

Induction: If $m_{n-1} = 2^{n-1} - 1$, then

$$\begin{aligned} m_n &= 2m_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \\ &= 2^n - 1 \end{aligned}$$

Rabbit Population Growth and Fibonacci Numbers

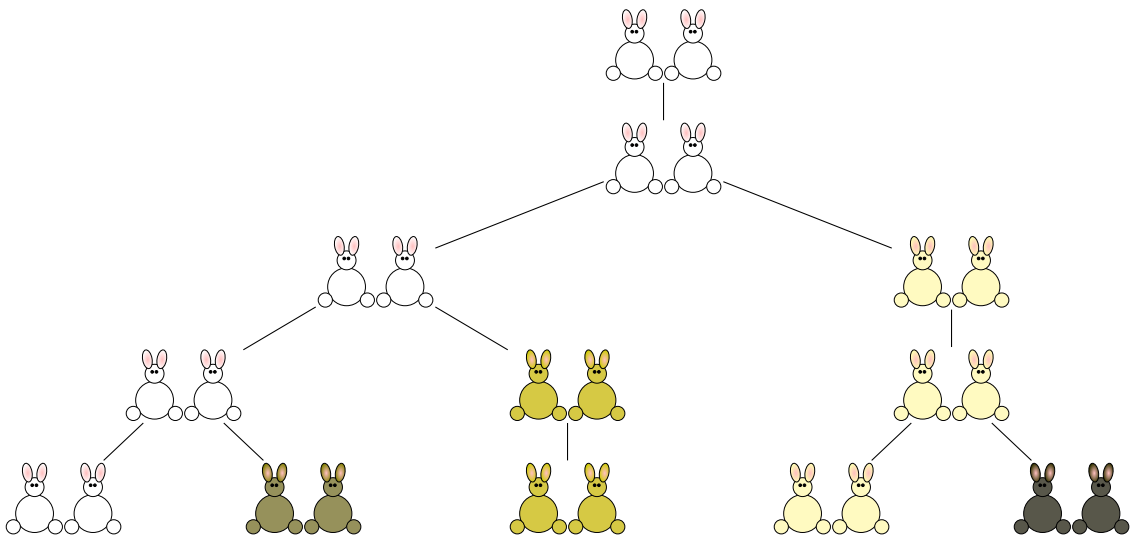
Overrun by rabbits, Fibonacci determined to model their growth. He let F_n name the number of pairs of rabbits alive during month n. Fibonacci's rules for bunny growth are:

1. In the beginning there were no pairs of rabbits: $F_0 = 0$.
2. At the first instance of bunny time, God created an original pair of rabbits: $F_1 = 1$.
3. At one month of age the pair mated: $F_2 = F_1 = 1$.
4. At two months of age they produced a pair of offspring: $F_3 = F_2 + 1 = 2$
5. Each pair repeated the same cycle every month.
6. No pair every perishes.

From these facts we can deduce that the number of pairs alive during cycle n is the sum of pairs alive during cycle $n - 1$ plus pairs alive during cycle $n - 2$ all of whom produce a pair of off-spring. That is,

$$F_n = F_{n-1} + F_{n-2}$$

Bunny time	0	1	2	3	4	5	6
Original pair		1	1	1	1	1	1
First Generation				1	1	1	1
Second Generation					1	1	1
Third Generation						2	2
Fourth Generation						0	3
Totals	0	1	1	2	3	5	8



The sequence of numbers generated by this scheme are called Fibonacci numbers. Fibonacci numbers can be computed by the recurrence equation

$$F_n = F_{n-1} + F_{n-2}$$

with initial condition $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers can also be computed by the function

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$$

where

$$\varphi = (1 + \sqrt{5})/2 \approx 1.618033$$

is the golden ratio and

$$\overline{\varphi} = \frac{1 - \sqrt{5}}{2} \approx -0.618033$$

is its conjugate root.

This is STRONGER form of recurrence. It requires an assignment of values to two initial condition: $F_0 = 0$ and $F_1 = 1$.

φ and $\overline{\varphi}$ are the zeros of the polynomial equation

$$x^2 - x - 1 = 0$$

In particular,

$$\varphi^2 = \varphi + 1$$

and

$$\overline{\varphi}^2 = \overline{\varphi} + 1$$

Basis: For $n = 0$, $F_0 = 0$ is given, and the function $\frac{\varphi^0 - \overline{\varphi}^0}{\sqrt{5}} = 0$ also.

Because of the stronger recurrence we must prove a STRONGER basis for induction. That is, for $n = 1$, $F_1 = 1$ is given, and the function $\frac{\varphi^1 - \overline{\varphi}^1}{\sqrt{5}} = 1$ also.

Induction: If $F_{n-2} = \frac{\varphi^{n-2} - \overline{\varphi}^{n-2}}{\sqrt{5}}$ and $F_{n-1} = \frac{\varphi^{n-1} - \overline{\varphi}^{n-1}}{\sqrt{5}}$, then

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &= \frac{\varphi^{n-1} - \overline{\varphi}^{n-1}}{\sqrt{5}} + \frac{\varphi^{n-2} - \overline{\varphi}^{n-2}}{\sqrt{5}} \\ &= \frac{\varphi^{n-2}(\varphi + 1) - \overline{\varphi}^{n-2}(\overline{\varphi} + 1)}{\sqrt{5}} \\ &= \frac{\varphi^{n-2}(\varphi^2) - \overline{\varphi}^{n-2}(\overline{\varphi}^2)}{\sqrt{5}} \\ &= \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} \end{aligned}$$

Induction on Arithmetic and Geometric Sequences

Summations occur in many applications. Sums over an arithmetic sequence are common. An arithmetic sequence has terms given by the function

$$a_n = mn + b \quad \text{for } n \in \mathbb{Z}$$

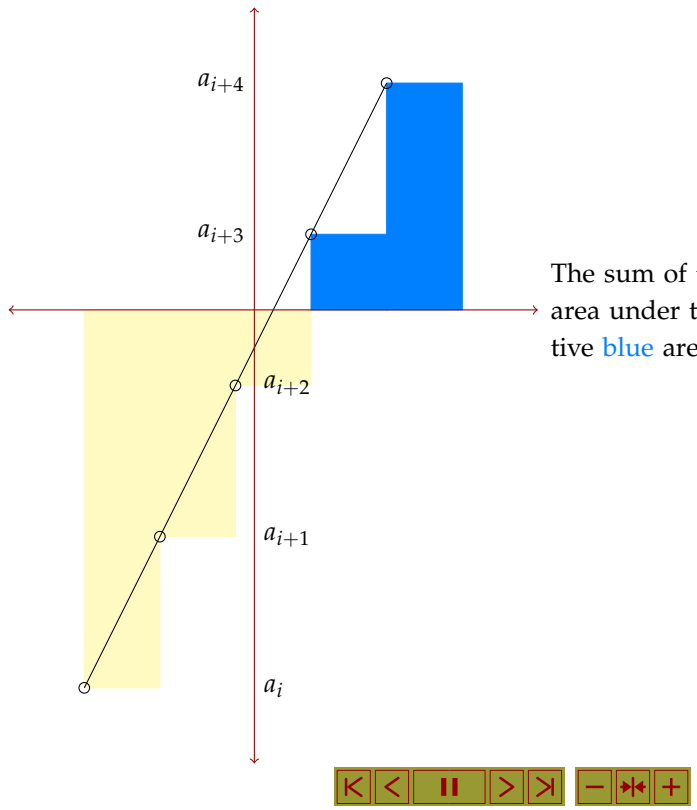
The terms a_n in an arithmetic sequence lie along a straight line with slope m and y-intercept b.

- A linear function is given by the formula

  $$y = f(x) = mx + b$$

  where

  - m is called the slope of the line; and
  - b is the y-intercept



The sum of terms is the negative tan area under the x-axis plus the positive blue area above the x-axis.

Let $t(i, n, m, b)$ denote the sum of a_k as k goes from i to $i + n - 1$ in increments of 1.

$$t(i, n, m, b) = \sum_{i \le k < i+n} a_k = \sum_{i \le k < i+n} (mk + b)$$

This sum is computed by the function

$$\begin{aligned} t(i, n, m, b) &= n \left[\frac{a_i + a_{i+n-1}}{2} \right] \\ &= n \left[\frac{[mi + b] + [m(i + n - 1) + b]}{2} \right] \\ &= n(mi + b) + \frac{mn(n-1)}{2} \end{aligned}$$

The formula for $t(i, n, m, b)$ looks formidable, but it is just the number of terms n times the average of the first and last term in the sum. The sum of the first n natural numbers $t(0, n, 1, 0) = n(n-1)/2$ is an instance of the general formula. These triangular numbers occurs by setting $i = 0$, $m = 1$ and $b = 0$.

That this function correctly computes the value of the sum can be established by mathematical induction on n, the number of terms in the sum.

Basis: For $n = 0$, the sum is over the empty range $i \le k < i$. This is the empty sum, and its value is defined to be 0.

It can be seen that when $n = 0$, the formula $t(i, 0, m, b) = 0$. This establishes the basis for induction.

Induction: If

$$t(i, n, m, b) = n(mi + b) + \frac{mn(n-1)}{2}$$

then

You deserve an A in algebraic manipulation if you follow this reasoning and can recreate it yourself.

$$\begin{aligned}
t(i, n+1, m, b) &= t(i, n, m, b) + [m(i+n) + b] \\
&= n(mi + b) + \frac{mn(n-1)}{2} + [m(i+n) + b] \\
&= (n+1)(mi + b) + \frac{mn(n-1)}{2} + mn \\
&= (n+1)(mi + b) + \frac{mn(n-1)}{2} + \frac{2mn}{2} \\
&= (n+1)(mi + b) + \frac{mn(n-1) + 2mn}{2} \\
&= (n+1)(mi + b) + \frac{mn(n+1)}{2}
\end{aligned}$$

This establishes the form of the function is preserved when n is incremented by 1.

A geometric sequence has terms given by the function

$$g_n = br^n \quad \text{for } r \neq 0 \text{ and } n \in \mathbb{Z}$$

The sum of g_k for k incrementing by 1 from i to $i + n - 1$ is

$$g(i, n) = \sum_{i \leq k < i+n} g_k = \sum_{i \leq k < i+n} br^k$$

It can be discovered that this sum can be computed by the function

$$g(i, n) = br^i \left[\frac{r^n - 1}{r - 1}\right] = b\left[\frac{r^{n+i} - r^i}{r - 1}\right]$$

That this function is correctly computes the value of the sum can be established by mathematical induction on n, the number of terms in the sum

Basis: For $n = 0$, the sum is over the empty range $i \leq k < i$. This is the empty sum, and its value is defined to be 0.

It can be seen that when $n = 0$, the formula $g(i, 0) = 0$. This establishes the basis for induction.

Induction: If

$$g(i, n) = br^i \frac{r^{n-i} - 1}{r - 1}$$

then

$$\begin{aligned}
g(i,\, n+1) &= g(i,n) + br^{i+n} \\
&= br^i \frac{r^n - 1}{r-1} + br^{i+n} \\
&= br^i \left(\frac{r^n - 1}{r-1} + r^n \right) \\
&= br^i \left(\frac{r^n - 1}{r-1} + \frac{r^n(r-1)}{r-1} \right) \\
&= br^i \left(\frac{r^n - 1 + r^{n+1} - r^n)}{r-1} \right) \\
&= br^i \left(\frac{r^{n+1} - 1}{r-1} \right)
\end{aligned}$$

This establishes the form of the function is preserved when n is incremented by 1.

Problems on Induction

1. Use mathematical induction to prove the sum of the first n natural numbers is $n(n-1)/2$. That is,

$$\sum_{0 \le k < n} k = \frac{n(n-1)}{2}$$

2. Use mathematical induction to prove the sum of the first n powers of 2 is $2^n - 1$. That is,

$$\sum_{0 \le k < n} 2^k = 2^n - 1$$

3. Use mathematical induction to prove the sum of the even natural numbers from 0 to $2n$ is $n(n+1)$. That is,

$$\sum_{k=0}^{n} 2k = n(n+1)$$

4. Use mathematical induction to prove the sum of products consecutive pairs of natural numbers is the product of three consecutive number divided by 3, that is,

$$\sum_{k=0}^{n-1} k(k-1) = \frac{n(n-1)(n-2)}{3}$$

5. Use mathematical induction to prove the summation formula

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

is true for all natural numbers $n \ge 0$.

6. Use mathematical induction to prove the sum the cubes is a square, that is,

$$\sum_{k=0}^{n-1} k^3 = \left(\frac{n(n-1)}{2}\right)^2$$

7. Use mathematical induction to prove the sum of k times $k!$ from $k = 0$ to $k = n - 1$ is $n! - 1$

$$\sum_{k=0}^{n-1} k \cdot k! = n! - 1.$$

8. Use mathematical induction to prove the summation formula

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$$

is true whenever n is a natural number greater than 1.

9. Prove 6 divides $7^n - 1$ for all natural numbers n.

10. Prove $n! \geq 2^n$ for all natural numbers $n \geq 4$.

11. Prove the function $T(n) = 3^n - 2$ satisfies the recurrence equation

$$T_n = 3T_{n-1} + 4$$

with initial condition $T_0 = -1$

12. Prove the function $T(n) = \lg(n)$ satisfies the recurrence equation

$$T_{2n} = T_n + 1$$

with initial condition $T_1 = 0$.

13. Use mathematical induction to prove that $s_n = 2^n + 3^n$ solves $s_n = 5s_{n-1} - 6s_{n-2}$ with initial conditions $s_0 = 2$, $s_1 = 5$.

14. Let F_n denote a term in the Fibonacci sequence

n	0	1	2	3	4	5	6	7	8	9	...
F_n	0	1	1	2	3	5	8	13	21	34	...

(a) Use mathematical induction to show that the sum of the first n Fibonacci numbers is a Fibonacci number minus one. Specifically, show that

$$F_0 + F_1 + \cdots + F_{n-1} = F_{n+1} - 1$$

(b) Use mathematical induction to prove that

$$F_n \leq 2F_{n-1} \quad \text{for } n \geq 2.$$

(c) Use induction over k to prove that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

(d) Prove that for $n \geq 1$

$$\gcd(F_n, F_{n-1}) = 1$$

(e) Prove that the sum of the first $n-1$ Fibonacci numbers is 1 less than the $n+1$ Fibonacci number. That is,

$$F_n = 1 + \sum_{k=0}^{n-2} F_k \quad \text{if } n \geq 2$$

(f) Prove that the sum of the odd-indexed Fibonacci numbers up to F_{2n+1} equals F_{2n+2}. That is,

$$\sum_{k=0}^{n} F_{2k+1} = F_{2n+2}$$

(g) Prove that

$$\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$$

(h) Prove that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{0}{n} = \sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

(i) Prove Cassini's identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad \text{for } n > 0$$

where F_k is a Fibonacci numbers number.

(j) Define the Lucas numbers L_n by

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n > 1, \quad L_0 = 2,\ L_1 = 1$$

i. Show that the Lucas numbers L_n satisfy the equation

$$L_n = F_{n+1} + F_{n-1}$$

ii. Prove that

$$F_{2n} = F_n L_n$$

where F_n is a Fibonacci number.

15. Prove that the sum of terms in a column of Pascal's triangle equals a term in the next column, that is,

$$\sum_{k=m}^{n-1} \binom{k}{m} = \binom{n}{m+1}, \quad n \geq m+1$$

16. Prove that the sum of term in a row of Pascal's triangle equals a power of 2, that is,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

17. Use mathematical induction to prove that the Harmonic solve the recurrence equation

$$\sum_{k=1}^{n-1} H_k + n = nH_n$$