CSE 1400 Applied Discrete Mathematics Predicates

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Predicate Logic

A Boolean statement is either `True` or `False`. By contrast, the truth value of a predicate statement depends on the value one or more variables, and functions and relations between them.

Predicate logic is also called first-order logic.

- A predicate statement can be `True` for all values of the variables. Logicians call this a UNIVERSAL AFFIRMATIVE. An instance of a universally `True` predicate statement is

  "For all natural number, the sum of the first n is $n(n-1)/2$."

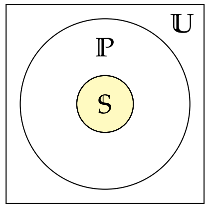
  Call "The sum (of the first n natural numbers)" the SUBJECT and label it $\mathbb{S}$. You can think of $\mathbb{S}$ as the set of numbers that are sums of first n natural numbers

  $$\mathbb{S} = \{0, 1, 3, 6, 10, 15, \ldots\}$$

  Call "$n(n-1)/2$" the PREDICATE and label it $\mathbb{P}$. You can think of $\mathbb{P}$ as the set of numbers that are of the form $n(n-1)/2$ where n is a natural number

  $$\mathbb{P} = \left\{ \frac{n(n-1)}{2} : n \in \mathbb{N} \right\}$$

  The predicate statement above can be represented by the Euler diagram



which shows every $s \in \mathbb{S}$ is a member of $\mathbb{P}$.

A universally affirmative predicate statement can be represented by the logic formula

$$(\forall s)(s \in \mathbb{S} \to s \in \mathbb{P})$$

which states that for all values s that are the sums of the natural numbers (starting at 0 and stopping at some finite value n) can be computed by the formula $n(n-1)/2$ for some natural number n.

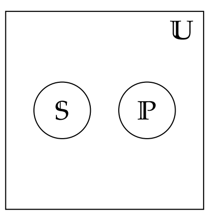
There are, of course, others ways to express this, for instance,

$$(\forall s \in \mathbb{S})(\exists n \in \mathbb{N})(s = n(n-1)/2)$$

- A predicate statement can be `True` for no values of the variables. Logicians call this a UNIVERSAL NEGATIVE. An instance of a universally `False` predicate statement is

> "For all natural numbers is the sum of the first n natural numbers is not equal to 7."

This predicate statement can be represented by the Euler diagram



which shows no element $s \in \mathbb{S}$ is a member of $\mathbb{P} = \{7\}$ and vice versa.

A universally negative statement can be written as

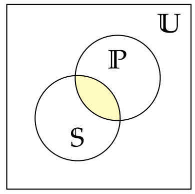
$$(\forall s)(s \in \mathbb{S} \to s \notin \mathbb{P})$$

which states if s is the sum of natural numbers (starting at 0 and stopping at some finite value n) then s is not equal to 7.

- A predicate statement can be `True` for at least one list of values for the variables. Logicians call this a PARTICULAR AFFIRMATIVE. An instance of predicate statement that is `True` for some values of the variable is

"The sum of the first n natural numbers is 10."

This predicate statement can be represented by the Euler diagram



which shows there is a non-empty intersection between $\mathbb{S}$ and $\mathbb{P} = \{10\}$.

A particular affirmative statement can be written as

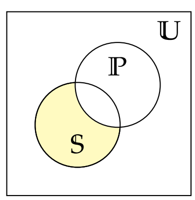
$$(\exists s)(s \in \mathbb{S} \wedge s \in \mathbb{P})$$

which states there is an element s that is in the intersection of $\mathbb{S}$ and $\mathbb{P}$ (the intersection is not empty).

- A predicate statement can be `False` for at least one list of values for the variables. Logicians call this a particular negative. An instance of predicate statement that is `False` for some values of the variable is

  "The sum of the first n natural numbers is not odd"

  This predicate statement can be represented by the Euler diagram



  which shows there is at least one element in $\mathbb{S}$ that is not in $\mathbb{P} = \{0, 2, 4, \ldots\}$.

  A particular negative statement can be written as

$$(\exists s)(s \in \mathbb{S} \wedge s \notin \mathbb{P})$$

  which states there is an element s that is in $\mathbb{S}$ and not in $\mathbb{P}$.

In each of the above predicate statements grammarians call "The sum (of the first n natural numbers)" the SUBJECT and they call "$n(n-1)/2$", "7", "10", "not odd" the PREDICATES.

The following statements are instances of predicate statements. The statements are written in English and mathematical notation. Some of the statements are `True` for all values of their variables, some are `True` for some values of their variables, and some are never `True`.

1. The sum of the first n natural numbers is $n(n-1)/2$.

$$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$$

2. The sum of the first n powers of 2 is $2^n - 1$.

$$\sum_{k=0}^{n-1} 2k = 2^n - 1$$

3. The sum of the first n odd natural numbers is n squared.

$$\sum_{k=0}^{n-1} (2k+1) = n^2$$

4. The integer n times $(n+2)$ is an even number.

$$n(n+2) \equiv 0 \pmod 2$$

5. If $n \geq 10$, then $2^n < n^3$.

$$(n \geq 10) \to (2^n < n^3)$$

6. If $a, n \in \mathbb{Z}$ and $n \neq 0$, then there exists integers q and r, called the quotient and remainder such that

$$a = q \cdot n + r$$

and $0 \leq |r| < |n|$.

$$(a, n \in \mathbb{Z} \wedge n \neq 0) \to (\exists q, r \in \mathbb{Z} \ni a = q \cdot n + r \wedge 0 \leq r < |n|)$$

7. The real number x squared minus 1 is equal to 0.

$$x^2 - 1 = 0$$

8. The real number x is greater than real number y.

$$x > y$$

9. Given a set with n elements, there are $n!/(k!(n-k)!)$ different subsets with k elements.

Being lazy, mathematicians use shorthand notation for predicates. For instance, $\text{P}(n)$ can stand for the predicate

> "The sum of the first n natural numbers is $n(n-1)/2$."

and $\text{G}(x, y)$ can stand for the predicate

"The real number x is greater than real number y."

Given a predicate statement $\text{P}(n)$, we would like to determine if it is universally `True`, universally `False`, or `True` in some instances and `False` in others.

1. A universally affirmative predicate $\text{P}(n)$, one that is `True` for all values of its variable(s), can be written as

$$(\forall n)(\text{P}(n))$$

2. A universally negative predicate $\text{P}(n)$, one that is `True` for no values of its variable(s); that is, `False` for all values of its variable(s), can be written as

$$\neg(\exists n)(\text{P}(n))$$

or

$$(\forall n)(\neg \text{P}(n))$$

3. A particular affirmative predicate $\text{P}(n)$, one that is `True` for some values of its variable(s) can be written as

$$(\exists n)(\text{P}(n))$$

4. A particular negative predicate $\text{P}(n)$, one that is `False` for some values of its variable(s) can be written as

$$(\exists n)(\neg \text{P}(n))$$

The phrases "for all" and "for some" are called quantifiers. Mathematicians use the symbol $\forall$ to stand for "for all." They use the symbol $\exists$ to stand for "for some."

PREDICATE LOGIC REASONS OVER COLLECTIONS OF THINGS . This contrasts with Boolean logic, which reasons over a single instance. Predicate logic adds two QUANTIFIERS to propositional logic. The quantifiers are SOMETHING and EVERYTHING, as in SOME THING satisfies a property or EVERY THING satisfies a property. For instance, you may say that THERE IS a real number x that satisfies the equation

$$x^2 - x - 1 = 0$$

There are, in fact, two solutions. The golden ratio $\varphi = (1 + \sqrt{5})/2$ and its conjugate $\overline{\varphi} = (1 - \sqrt{5})/2$.

Or, you may say that EVERY real number x satisfies the equation

$$x + 0 = x$$

Mathematicians use standard symbols for these and similar phrases.

$\exists$ "There exists" "There is a least one"

The symbol $\exists!$ is used to say "there is one and only one," that is, the THING that exists is UNIQUE.

$\forall$ "For all" "For every"

Quantifiers modify a PREDICATE, which is a statement about a THING . If the name of the THING is x and the name of the statement is P, then the predicate has the form

$$\text{P}(x)$$

Statements in predicate logic have forms such as

All S are P

No S are P

Some S are P

Some S are not P

Predicates arise in computing as Boolean expressions that control the flow of a program. Predicates state relations between functions, variables and constants. For instance,

- The point (x, y) lies on the line determined by parameters a, b, and c can be written
$$ax + by = c$$
and named $\text{P}(x, y, a, b, c)$.
- That a is congruent to b mod n can be written
$$a - b = cn$$
and named $\text{P}(a, b, n, c)$.
- That $\mathbb{X}$ is a subset of $\mathbb{Y}$ can be written
$$(x \in \mathbb{X}) \to (y \in \mathbb{Y})$$
and named $\text{P}(x, \mathbb{X}, y, \mathbb{Y})$.
- That f is a one-to-one function can be written
$$(\forall x_0, x_1 \in \mathbb{X})(x_0 \neq x_1)) \to (f(x_0) \neq f(x_1))$$
and named $\text{P}(x_0, x_1, f)$.
- That f is an onto function can be written
$$(\forall y \in \mathbb{Y})(\exists x \in \mathbb{X})(f(x) - y)$$
and named $\text{P}(x, y, f)$.

In a given instance the predicate may be `True` or `False`. Describing the set of values where a predicate is `True` leads to useful definitions and theorems.

Let $a = 2$, $b = 3$ and $c = 7$ in $ax + by = c$. Then $2 \cdot 1 + 3 \cdot 5 = 7$ is `False` but $2 \cdot 2 + 3 \cdot 1 = 7$ is `True`.

Line: The set of ALL points (x, y) in the Cartesian coordinate plane that satisfy the equation

$$ax + by = c$$

for GIVEN real-valued constants a, b, and c.

Congruence mod n: $a \equiv b \mod n$ if THERE EXISTS an integer c such that

$$a - b = cn$$

Subset: $\mathbb{X} \subseteq \mathbb{Y}$ if FOR ALL $x \in \mathbb{X}$, it is the case that $x \in \mathbb{Y}$.

One-to-one function: When every pair of distinct domain values map to distinct range values, the function f is one-to-one.

Here are two conjectures that appear to be `True`, but no one has discovered a proof or counterexample.

Twin primes: There are infinitely many prime pairs $(p, p+2)$ For instance twin prime pairs include (3, 5), (5, 7), (11, 13), and (17, 19).

$$(\forall n \in \mathbb{N})(\exists p \in \mathbb{P})((p > n) \wedge (p, p+2 \in \mathbb{P}))$$

Goldbach: Every even integer greater than 2 can be written as the sum of two primes. For instance, $4 = 2+2$, $6 = 3+3$, $8 = 5+3$, and $10 = 5+5$.

$$(\forall n \in \mathbb{N})[(n > 2) \wedge (n \mod 2 = 0) \implies (\exists p \in \mathbb{P})(\exists q \in \mathbb{P})(p, q \in \mathbb{P} \wedge n = p+q)]$$

In computing practice, the sets of THINGS reasoned about in predicate statements belong to some type.

- athletes
- mammals
- natural numbers
- green things
- logicians
- rock bands
- functions on the integers
- permutations
- floating point numbers

Predicate Logic Inference Rules

THERE ARE FOUR BASIC RULES TO REASON ABOUT QUANTIFIED STATEMENTS.

Universal Instantiation If $(\forall x \in \mathbb{U})(\text{P}(x))$ is `True` then $(\exists c \in \mathbb{U})(\text{P}(c))$ is `True`

Universal Generalization If $\text{P}(c)$ is `True` for an arbitrary $c \in \mathbb{U}$, then $(\forall x \in \mathbb{U})(\text{P}(x))$ is `True`

Existential Instantiation If $(\exists x \in \mathbb{U})(\text{P}(x))$ is `True`, then $\text{P}(c)$ is `True` for some instance c

Existential Generalization If $\text{P}(c)$ is `True`, then $(\exists x \in \mathbb{U})(\text{P}(x))$ is `True`

The fact that the square of every integer is greater than or equal to 0 implies $4^2 \geq 0$.
The fact that the square of an even integer is divisible by four implies $(\forall 2n \in \mathbb{N})(n \in \mathbb{N})(4 \operatorname{div} n^2)$.

Negation of Quantified Predicate Statements

IF SOMETHING IS NOT ALWAYS `True`, THERE IS AN INSTANCE WHERE IT IS `False`.

$$\neg[(\forall x \in \mathbb{X})(\text{P}(x))] \equiv (\exists x \in \mathbb{X})(\neg\text{P}(x))$$

If there is not an instance when something `True`, then the something is always `False`.

$$\neg[(\exists x \in \mathbb{X})(\text{P}(x))] \equiv (\forall x \in \mathbb{X})(\neg\text{P}(x))$$

Reasoning about Quantification Order

WHEN IS IT THE CASE THAT THE ORDER OF $\forall$ AND $\exists$ CAN BE SWAPPED? Consider the three cases

1. $(\forall x \in \mathbb{X})(\forall y \in \mathbb{Y})(\mathrm{P}(x, y)) \stackrel{?}{\equiv} (\forall y \in \mathbb{Y})(\forall x \in \mathbb{X})(\mathrm{P}(x, y))$
2. $(\exists x \in \mathbb{X})(\exists y \in \mathbb{Y})(\mathrm{P}(x, y)) \stackrel{?}{\equiv} (\exists y \in \mathbb{Y})(\exists x \in \mathbb{X})(\mathrm{P}(x, y))$
3. $(\forall x \in \mathbb{X})(\exists y \in \mathbb{Y})(\mathrm{P}(x, y)) \stackrel{?}{\equiv} (\exists y \in \mathbb{Y})(\forall x \in \mathbb{X})(\mathrm{P}(x, y))$

In the first two questioned equivalences above, the repeated quantifiers are tied together using AND. Therefore, $\forall$ pairs can be commuted and $\exists$ pairs can be too.

For instance, "for every x and for every y $\mathrm{P}(x, y)$ is `True`" is equivalent to "for every y and for every x $\mathrm{P}(x, y)$ is `True`."
And "there is an x and there is a y $\mathrm{P}(x, y)$ such that is `True`" is equivalent to "there is an y and there is a x $\mathrm{P}(x, y)$ such that is `True`."
Consider the statement "All mathematicians love a logician." Does it mean "All mathematicians love Haskell Curry." a particular logician? Or does it mean "Every mathematician loves some logician." who may or may not be Haskell Curry?

The third questioned equivalence is read

> "Is 'for all x there is a y such that $\mathrm{P}(x, y)$ is `True`' equivalent to 'there is a y such that for every x $\mathrm{P}(x, y)$ is `True`'?"

Consider the `True` statement

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y = 0)$$

For any given x, let $y = -x$. Consider the `False` statement

$$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x + y = 0)$$

There is no fixed real number y satisfies the equation $x + y = 0$ for every x. Therefore, $\forall$ and $\exists$ pairs cannot be commuted.

Problems on Predicate Logic

1. Use mathematical notation to write the predicate "The weak can never forgive," that is attributed to Mahatma Gandhi,

   Answer: Let the domain be $\mathbb{P}$, the set of all people, and let $p \in \mathbb{P}$ represent a person. Let $\mathrm{W}(p)$ be the predicate "p is weak" and let $\mathrm{F}(p)$ be the predicate "p forgives." Gandhi's statement is $\mathrm{W}(p) \to \neg\mathrm{F}(p)$.

2. The predicate statements below come from (Carroll, 1958). Write them using mathematical notation.

   (a) No Frenchmen like plumpudding.
   Answer: $(\forall p \in \mathbb{P})(\mathrm{F}(p) \to \neg\mathrm{LP}(p))$, where $\mathrm{F}(p)$ is the predicate "p is a Frechman" and $\mathrm{LP}(p)$ is the predicate "p likes plumpudding. An equivalent statement is $(\forall p \in \mathbb{P})(\mathrm{LP}(p) \to \mathrm{F}(p))$.

   (b) All Englishmen like plumpudding.
   Answer: $(\forall p \in \mathbb{P})(\mathrm{E}(p) \to \mathrm{LP}(p))$.

   (c) Some thin persons are not cheerful.
   Answer: $(\exists p \in \mathbb{P})(\mathrm{T}(p) \land \neg\mathrm{C}(p))$.

   (d) All pigs are fat.
   Answer: $(\forall p \in \mathbb{PIG})(\mathrm{FAT}(p))$.

(e) Some lessons are difficult.
Answer: $(\exists l \in \mathbb{L})(\mathrm{D}(l))$.

(f) All clever people are popular.
Answer: $(\forall p \in \mathbb{P})(\mathrm{C}(p) \to \mathrm{P}(p))$.

(g) Some healthy people are fat.
Answer: $(\exists p \in \mathbb{P})(\mathrm{H}(p) \land \mathrm{FAT}(p))$.

3. Which of the statements below are `True` and which are `False`?

(a) $(\forall n \in \mathbb{N})(n^2 > 0)$
Answer: This is `False`; it fails to hold for $0 \in \mathbb{N}$.

(b) $(\exists n \in \mathbb{N})(n^2 > 0)$
Answer: This is `True`; it holds for $n = 1$.

(c) $(\forall n \in \mathbb{N})(n^2 \leq 0)$
Answer: This is `False`; it fails to hold for $1 \in \mathbb{N}$.

(d) $(\exists n \in \mathbb{N})(n^2 \leq 0)$
Answer: This is `True`; it holds for $n = 0$.

4. Let $\mathrm{P}(x)$ be the statement $x^2 = x + 1$ over the domain of real numbers. What is the truth value of the following statements.

(a) $\mathrm{P}(0)$
Answer: $\mathrm{P}(0)$ is `False`.

(b) $\mathrm{P}((1+\sqrt{5})/2)$
Answer: $\mathrm{P}(1+\sqrt{5})/2)$ is `True`.

(c) $(\forall x \in \mathbb{R})(\mathrm{P}(x))$
Answer: $(\forall x \in \mathbb{R})(\mathrm{P}(x))$ is `False`.

(d) $(\forall x \in \mathbb{R})(\neg\mathrm{P}(x))$
Answer: $(\forall x \in \mathbb{R})(\neg\mathrm{P}(x))$ is `False`.

(e) $(\exists x \in \mathbb{R})(\mathrm{P}(x))$
Answer: $(\exists x \in \mathbb{R})(\mathrm{P}(x))$ is `True`.

(f) $(\exists x \in \mathbb{R})(\neg\mathrm{P}(x))$
Answer: $(\exists x \in \mathbb{R})(\neg\mathrm{P}(x))$ is `True`.

5. This question is from (Papadimitriou, 1994). Let $\mathrm{CANFOOL}(p, t)$ be the proposition "You can fool person p at time t." Write Abraham Lincoln's famous quotation

> *You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time.*

using the quantifiers over people p and times t and the $\mathrm{CANFOOL}(p, t)$ predicate.
Answer: $(\exists p)(\forall t)(\mathrm{CANFOOL}(p, t)) \land (\forall p)(\exists t)(\mathrm{CANFOOL}(p, t)) \land \neg(\forall p)(\forall t)(\mathrm{CANFOOL}(p, t))$.

6. This question is from the MIT OpenCourseWare course on Discrete Mathematics. Suppose $S(n)$ is a predicate on natural numbers, n, and suppose

$$(\forall k \in \mathbb{N})(S(k) \to S(k+2)).$$

holds `True`, Are the statements below always `True`, always `False`, or sometimes `True` and sometimes `False`?

- Consider the statement
$$(\forall n \leq 100)(S(n)) \land (\forall n > 100)(\neg S(n))$$
Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
Answer: In this case, S is `True` for n up to 100 and `False` from 101 on. So $S(99)$ is `True`, but $S(101)$ is `False`. That means that $S(k) \not\to S(k+2)$ for $k = 99$. This case is impossible.

- Consider the statement
$$S(1) \to (\forall n)(S(2n+1))$$
Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
Answer: This assertion says that if $S(1)$ holds, then $S(n)$ holds for all odd n. This case is always `True`.
- Consider the statement
$$(\exists n)(S(2n) \to (\forall n)(S(2n+2)))$$
Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
Answer: If $S(n)$ is always `True`, this assertion holds. So this case is possible. If S(n) is true only for even ngreater than 4, (1) holds, but this assertion is false. So this case does not always hold.
- Consider the statement
$$(\exists n)(\exists m > n)(S(2n) \land \neg S(2m))$$
Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
Answer: This assertion says that S holds for some even number, $2n$, but not for some other larger even number, $2m$. However, if $S(2n)$ holds, we can apply n — m times to conclude $S(2m)$ also holds. This case is impossible.
- Consider the statement
$$(\exists n)(S(n)) \to (\forall n)(\exists m > n)(S(m))$$
Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
Answer: This assertion says that if S holds for some n, then for every number, there is a larger number, m, for which S also holds. Since (1) implies that if there is one n for which $S(n)$ holds, there are an infinite, increasing chain of k's for which $S(k)$ holds, this case is always true.