CSE 1400 Applied Discrete Mathematics
Recurrences

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Abstract

The definition of "recursive" that I like is

Recursive: adj. See: Recursive

"Devil's DP Dictionary,"
Stan Kelly-Bootle,
English computer scientist and author (1929 –)

Recurrence Equations

RECURRENCE EQUATIONS ARE THE DISCRETE ANALOG OF DIFFERENTIAL EQUATIONS. Recurrence equations have two parts.

Recurrence equations are similar to differential equations. Recurrence equations are studied in discrete mathematics, Differential equations studied in continuous mathematics.

1. An equation that describes how new values are computed from previous values. The form a recurrence equation is something like

$$t_n = \text{some formula involving } t_{n-1}$$
$$\text{and perhaps lower terms such as}$$
$$t_{n-2}, t_{n-3}, \ldots, t_0$$

2. One or more initial values to seed the computational process. These initial values are given for the first term t_0 and perhaps higher terms such as t_1 and t_2.

There are several basis recurrence equations that occur in discrete mathematics.

"Can you do Addition?" the White Queen asked.
"What's one and one and one and one and one and one and one and one and one and one?"
"I don't know," said Alice. "I lost count."
"She can't do Addition," the Red Queen interrupted.
"Can you do Subtraction? Take nine from eight."
A simple recurrence, but Alice is fundamental to counting.

The Alice Recurrence Equation

Alice terms are defined by the recurrence equation equation

$$a_n = a_{n-1} \quad \forall n \in \mathbb{N},\ n \geq 1$$

with INITIAL CONDITION $a_0 = 1$. The Alice recurrence generates terms in the unary sequence

$$\vec{A} = \langle 1, 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$$

Terms in the Alice sequence can also be computed by the constant function

$$a(n) = 1 \quad \forall n \in \mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$$

UNARY NOTATION for the natural numbers is based on the Alice sequence. Mathematical induction can be used to prove the function $a(n) = 1$ solves the recurrence equation

Using the unary alphabet $\{1\}$ with only the character 1 can name the positive natural numbers $1 = 1$ $2 = 11$, $3 = 111$, ... Of course, UNARY NOTATION wastes space. To write 256 in unary notation requires a string of 256 unary 1's, but only 3 digits or 9 bits.

$$a_n = a_{n-1}$$

with initial condition $a_0 = 1$.

Initial Condition: The function $a(n) = 1$ correctly computes the initial term $a_0 = 1$. That is, $a(0) = 1$ since $a(n) = 1$ for all $n \in \mathbb{N}$.

Induction Premise: Pretend that for some unspecified n, $a_{n-1} = 1$. We know this is true for $n = 1$.

Inductive Conclusion: Use the recurrence equation

$$a_n = a_{n-1}$$

and the premise $a_{n-1} = 1$, to conclude that $a_n = 1$.

The Gauss Recurrence Equation

Gauss terms are computed by the recurrence equation equation

$$g_n = g_{n-1} + 1 \quad \forall n \in \mathbb{N}, n \geq 1$$

with INITIAL CONDITION $g_0 = 0$. The Gauss recurrence generates terms in the sequence of natural numbers

$$\vec{G} = \langle 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots \rangle$$

Terms in the Gauss sequence can also be computed by the function

$$g(n) = n \quad \forall n \in \mathbb{N}$$

Mathematical induction can be used to prove the function $g(n) = n$ solves the recurrence equation

$$g_n = g_{n-1} + 1$$

with initial condition $g_0 = 0$.

Initial Condition: The function $g(n) = n$ correctly computes the initial term $g_0 = 0$.

Induction Premise: Pretend that for some unspecified n, $g_{n-1} = (n-1)$.

Inductive Conclusion: Using the recurrence equation

$$g_n = g_{n-1} + 1$$

and $g_{n-1} = (n-1)$, to conclude that $g_n = (n-1) + 1 = n$.

The well known story of Gauss summing the natural numbers from 1 to 100 and Gauss' genius at number theory caused me to name the SEQUENCE of natural numbers for Gauss.

The Triangular Recurrence Equation

Triangular terms are computed by the RECURRENCE equation

$$t_n = t_{n-1} + (n-1) \quad \forall n \in \mathbb{N}, n \geq 1$$

with INITIAL CONDITION $t_0 = 0$. The Triangular recurrence generates terms in the fully-connected numbers

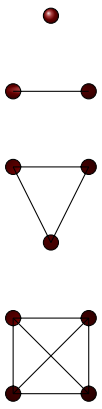
$$\vec{T} = \langle 0, 0, 1, 3, 6, 10, 15, 21, 28, 36, \ldots \rangle$$

Terms in the Triangular sequence can also be computed by the function

$$t(n) = \frac{n(n-1)}{2} \quad \forall n \in \mathbb{N}$$

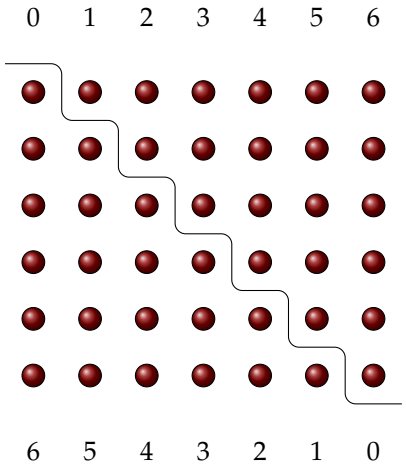
Summing terms from the Gauss sequence produces triangles.

Consider graphs with 1, 2, 3 and 4 nodes. They are fully-connected if there is an edge between each pair of nodes. The number of edges in such graphs is 0, 1, 3 and 6. This sequence of terms can start with 0, the number of edges in the empty graph.



Some authors define triangular numbers by the function $t(n) = n(n+1)/2$ and start with initial condition $n = 1$.

$$\sum_{0 \le k < 7} k = 0 + 1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{7(7-1)}{2}$$

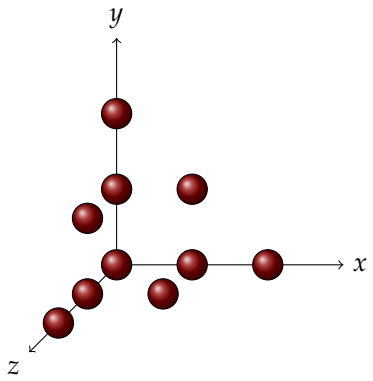


$$\sum_{0 \le k < 7} (6 - k) = 6 + 5 + 4 + 3 + 2 + 1 + 0 = 21 = \frac{7(7-1)}{2}$$

The triangular number t_n is the sum of the first n natural numbers.

$$\begin{aligned} t_n &= \sum_{0 \le k < n} k = 0 + 1 + \cdots + (n-1) + n = \frac{n(n-1)}{2} \\ &= \sum_{0 \le k < n} (n - 1 - k) = (n-1) + (n-2) + \cdots + 0 = \frac{n(n-1)}{2} \end{aligned}$$

You can stack pixelated triangles on top one another to create pyrimids.



The sum triangular numbers are called PYRAMIDAL or TETRAHEDRAL numbers. The sequence of PYRIMIDAL numbers is

$$\vec{Y} = \langle 0, 0, 0, 1, 4, 10, 20, 35, 56, 84, \ldots \rangle$$

where terms p_n in $\vec{Y}$ are computed as partial sums of terms in $\vec{T}$.

$$p_n = \sum_{0 \le k < n} \frac{k(k-1)}{2} = 0+0+1+3+\cdots+\frac{(n-1)(n-2)}{2} = \frac{n(n-1)(n-2)}{3 \cdot 2}$$

You could, if you like, stack pyrimid on pyramids in 4-dimensional space and create what are called PENTATOPE numbers, but let's stop here.

It is useful to write the Alice, Gauss, triangular, and higher order numbers as columns in a two-dimensional list.

Do you recognize Pascal's triangle in this array?

Alice	Gauss	Triangular	Pyrimidal	Pentatopic
1	0	0	0	0
1	1	0	0	0
1	2	1	0	0
1	3	3	1	0
1	4	6	4	1
1	5	10	10	5
1	6	15	20	15

The Mersenne Recurrence Equation

Mersenne terms are computed by the RECURRENCE equation

$$m_n = 2m_{n-1} + 1 \quad \forall n \in \mathbb{N}, n \ge 1$$

with INITIAL CONDITION $m_0 = 0$. Terms in the Mersenne sequence name the largest natural numbers that can be named binary notation and n bits.

The Mersenne number $2^n - 1$ is the largest natural number that can be named using n bits.

$$\vec{M} = \langle 0, 1, 3, 7, 15, 31, 63, 127, 255, \ldots \rangle$$

Terms in the Mersenne sequence are computed by the function

$$m(n) = 2^n - 1 \quad \forall n \in \mathbb{N}$$

```
mersenne :: Natural -> Natural
mersenne n = 2**n - 1
```

or

```
mersenne :: Natural -> Natural
mersenne 0 = 0
mersenne n = 2 * mersenne (n-1) + 1
```

Figure 1: Haskell function that computes Mersenne numbers.

Mathematical induction can be used to prove the function $m(n) = 2^n - 1$ solves the recurrence equation

$$m_n = 2m_{n-1} + 1$$

with initial condition $m_0 = 0$.

Initial Condition: The function $m(n) = 2^n - 1$ correctly computes the initial term $m_0 = 0$, that is, $m(0) = 0$.

Induction Premise: Pretend that for some unspecified n, $m_{n-1} = 2^{n-1} - 1$. We know this is true for $n = 1$.

Inductive Conclusion: Use the recurrence equation

$$m_n = 2m_{n-1} + 1$$

and the premise $m_{n-1} = 2^{n-1} - 1$ to write

$$\begin{aligned} m_n &= 2m_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \\ &= 2^n - 1 \end{aligned}$$

The largest known prime is the Mersenne number

$$2^{43112609} - 1 \quad \text{which has} \quad 12,978,189 \text{ digits}$$

The Mersenne numbers are a rich source for large prime numbers. You too can join the Great Internet Mersenne Prime Search (GIMPS) at http://www.mersenne.org/

The Factorial Recurrence Equation

Factorial terms are computed by the RECURRENCE equation

$$f_n = nf_{n-1} \quad \forall n \in \mathbb{N},\ n \geq 1$$

with INITIAL CONDITION $f_0 = 1$. Terms in the Factorial sequence count the number of permutations of n distinct THINGS .

The common notation is $n!$ read "n factorial."

$$\vec{n!} = \langle 1, 1, 2, 6, 24, 120, 720, 5040, \ldots \rangle$$

While there is no simple function that computes $n!$ Stirling's approximations

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

is a remarkably accurate formula for $n!$.

```
factorial :: Natural -> Natural
factorial 0 = 1
factorial n =
   n * factorial (n-1)
```

Figure 2: Haskell function that computes factorials.

The Fermat Recurrence Equation

Fermat terms are computed by several RECURRENCE equations

$$\begin{aligned} r_n &= (r_{n-1} - 1)^2 + 1 \\ r_n &= r_{n-1} + 2^{2^{n-1}} r_0 \cdots r_{n-2} \\ r_n &= r_{n-1}^2 - 2(r_{n-2} - 1)^2 \\ r_n &= r_0 \cdots r_{n-1} + 2 \end{aligned}$$

with INITIAL CONDITION $r_0 = 3$. The solution to each of these recurrences is the function

Terms in the Fermat sequence are 1 more than the count of Boolean functions on n Boolean variables. That is, the Fermat sequence is important because it closely counts the number of different Boolean functions in n variables.

$$r(n) = 2^{2^n} + 1 \quad \forall n \in \mathbb{N}$$

The sequence of Fermat terms is

$$\vec{R} = \langle 3, 5, 17, 257, 65\,537, 4\,294\,967\,297, \ldots \rangle$$

That $r(n) = 2^{2^n} + 1$ solves each of the recurrences above can be demonstrated by mathematical induction.

Initial Condition: The function $r(n) = 2^{2^n} + 1$ correctly computes the initial term $r_0 = 3$, that is, $r(0) = 2^{2^0} + 1 = 2^1 + 1 = 3$.

Induction Premise: Pretend that for some unspecified n, $r_{n-1} = 2^{2^{n-1}} + 11$. We know this is true for $n = 1$.

Inductive Conclusion: Use the recurrence equation

$$r_n = (r_{n-1} - 1)^2 + 1$$

and the premise $r_{n-1} = 2^{2^{n-1}} + 1$ to write

$$\begin{aligned} r_n &= (r_{n-1} - 1)^2 + 1 \\ &= (2^{2^{n-1}})^2 + 1 \\ &= 2^{2^{n-1} \cdot 2} + 1 \\ &= 2^{2^n} + 1 \end{aligned}$$

The Fibonacci Recurrence Equation

Terms in the Fibonacci sequence can computed by the recursion

$$F_n = F_{n-1} + F_{n-2}, \quad n \in \mathbb{N}, n \geq 2$$

with INITIAL CONDITIONS $F_0 = 0$ and $F_1 = 1$.

$$\vec{F} = \langle 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \rangle$$

The golden ratio $\varphi = (1 + \sqrt{5})/2 \approx 1.618033$ and its conjugate $\overline{\varphi} = (1 - \sqrt{5})/2 \approx -0.618033$ can be used to write Fibonacci numbers as

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$$

```
fibonacci :: Natural -> Natural
fibonacci 0 = 0
fibonacci 1 = 1
fibonacci (n) = fibonacci (n-1)
   + fibonacci(n-2)
```

Figure 3: Haskell function that computes Fibonacci numbers.

Since

$$\lim_{n \to \infty} \overline{\varphi}^n = 0$$

The Fibonacci number F_n is approximately $1.6F_{n-1}$ which makes the Fibonacci sequence a good way to translate from miles to kilometers: One mile is about 1.6 kilometers. For instance, 30 miles written in Fibonacci numbers is $30 = 21 + 8 + 1$ which is approximately $34 + 13 + 2 = 49$ kilometers.

Still More Recurrences

Bisection

There are several problems that can be solved by one "test" that narrows the solution space by one-half. This bisection solution technique can be described by the recurrence equation

$$b_{2n} = b_n + 1$$

which can be interpreted as stating that "to solve a problem of size $2n$, do 1 thing and then solve a problem of size n."

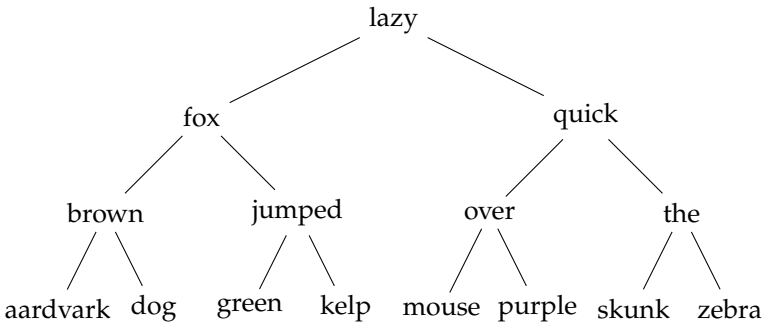
For instance to find the word "mouse" in the sorted list

aarvark, brown, dog, fox, green, kelp, jumped, lazy, mouse, over, quick, purple, skunk, the, zebra

Dictionaries, telephone books, and other look-up tables are sorted to enable quick searches.

- Compare "mouse" to "lazy," the middle word in the list.
- Seeing they are not equal eliminates half the words to be searched.
- Then compare "mouse" to "quick" the middle word of the last half.
- Seeing they are not equal eliminates the upper quarter the words to be searched.

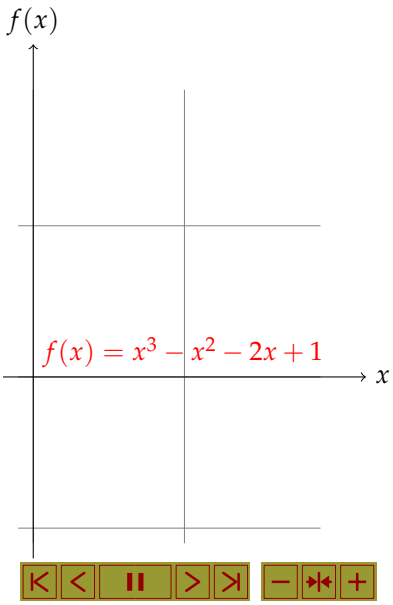
Continuing in this manner locates "mouse" in 4 comparisons.



In a similar manner, the zero of a function can be located by the bisection rootfinding method. For instance, the function $f(x) = x^3 - x^2 - 2x + 1$ changes sign from $x = 0$ to $x = 1$.

1. Evaluating $f(x)$ at $x = 0.5$ allows the interval $[0.5, 1.0]$ to be discarded.
2. Evaluating $f(x)$ at $x = 0.25$ allows the interval $[0.0, 0.25]$ to be discarded.
3. Evaluating $f(x)$ at $x = 0.375$ allows the interval $[0.25, 0.375]$ to be discarded.
4. Evaluating $f(x)$ at $x = 0.4375$ allows the interval $[0.375, 0.4375]$ to be discarded.

In only 4 steps the search domain has been reduced from length 1 to length 0.0625.



The logarithmic function

$$b_n = \lg n$$

satisfies the recurrence equation $b_{2n} = b_n + 1$. This can be verified by substitution into the equation.

$$\begin{aligned} b_{2n} &= \lg 2n \\ &= \lg 2 + \lg n \\ &= 1 + \lg n \\ &= 1 + b_n \end{aligned}$$

Newton's Rootfinding Method

Newton devised a fast method to compute roots or zeros of a function $f(x)$. A root of $f(x)$ is a value of x such that $f(x) = 0$. Newton's method is to

1. Guess an initial value x_0 to be a root of $f(x)$.
2. Construct the sequence

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \tag{1}$$

   where $f'(x)$ is the derivative of $f(x)$.

One application of Newton's method is the computation of square roots. For instance, the square root of 2 is a zero of the function $f(x) = x^2 - 2$. That is, $\sqrt{2}$ ia a root of the equation

$$x^2 - 2 = 0$$

$\sqrt{2}$ In this case, $f'(x) = 2x$, so that substituting into equation 1 yields the recurrence equation

$$\begin{aligned} x_n &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{2x_{n-1}^2}{2x_{n-1}} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{x_{n-1}^2}{2x_{n-1}} + \frac{2}{2x_{n-1}} \\ &= \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right) \end{aligned}$$

Starting with an initial guess that $x_0 = 1$ is equal to $\sqrt{2}$ the recurrence generates the sequence of approximations to $\sqrt{2}$

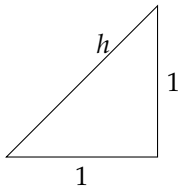
$$\left\langle 1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots \right\rangle$$

or in fixed-point notation

$$\langle 1.0,\ 1.5,\ 1.4666\cdots,\ 1.41421568627\cdots, \ldots \rangle$$

which is accurate to five decimal places.

Although Newton's name is attached this method the basic idea was known to ancient mathematicians of Mesopotamia, the region of modern day Iraq and Iran. Clay tablets dated from around 1800 B.C. to 1600 B.C. have been found there that show how to compute $\sqrt{2}$. These ancients considered an isosceles right triangle with legs of length 1



We know from the Pythagorean theorem that

$$1^2 + 1^2 = h^2$$

so that the hypotenuse h has length $h = \sqrt{2} \approx 1.41421356\cdots$ The Babylonians' knew how to approximate the value of $h = \sqrt{2}$ to many decimals places, although they used sexagesimal notation. Although it is not certain what reasoning the Babylonians' used, their calculations indicate this is what they did:

- Let $h_0 = 1$ be an initial approximation to $\sqrt{2}$
- Clearly 1 is too small as the Babylonians' could easily measure
- If $1 = \sqrt{2}$, then $1 \cdot 1 = \sqrt{2} \cdot \sqrt{2} = 2$ and $2/1 = \sqrt{2}$
- As it is, $2/1$ is too large
- The average of the under estimate 1 and the over estimate $2/1$ provides a better approximation to $\sqrt{2}$, call this

$$h_1 = \frac{1}{2}\left(h_0 + \frac{2}{h_0}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$$

- But $h_1 = 3/2$ is too large as the Babylonians' could measure
- If $3/2$ were the exact square root, then $2/(3/2) = 4/3$ would equal $\sqrt{2}$
- As it is, $4/3 \approx 1.333\cdots$ is too small
- The average of the over estimate $3/2$ and the under estimate $4/3$ will provide a better approximation

$$h_2 = \frac{1}{2}\left(h_1 + \frac{2}{h_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12} \approx 1.41166\cdots$$

- But $h_2 = 17/12$ is too small
- The Babylonians' carried out this iteration a few more times computing the $\sqrt{2}$ accurately to at least 9 decimal places
- That is, they next computed the average of $h_2 = 17/12$ and $2/h_2 = 24/17$

$$\begin{aligned} h_3 &= \frac{1}{2}\left(h_2 + \frac{2}{h_2}\right) \\ &= \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) \\ &= \frac{1}{2}\left(\frac{289 + 288}{17 \times 12}\right) \\ &\approx 1.41421568628\cdots \end{aligned}$$

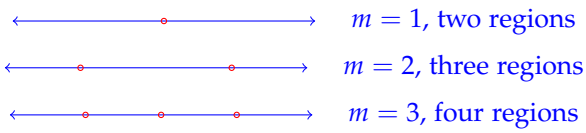
Newton generalized this idea and his method is one of the best rootfinding methods ever devised.

Dissecting Space with Hyperplanes

Consider this problem: Given an infinite straight line stretching in both directions

what is the maximum[1] number of regions the line can be cut into by using m points?

Using $m = 1$, $m = 2$, and $m = 3$ points, it is fairly clear that 2, 3, and 4 regions are the most there are.



And from these examples confidently conclude that when m points are used, the maximum number of regions is $m + 1$. This can be recorded in a table

Points	1	2	3	4	5	$\cdots$	m
Regions	2	3	4	5	6	$\cdots$	$m+1$

A more challenging problem is: What is the maximum number of regions the Euclidean plane can be cut into by m straight lines?

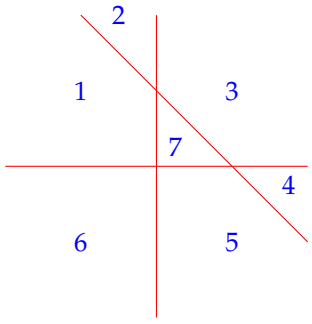
It is fairly clear that

When $m = 0$ the plane is undivided and there is 1 region

When $m = 1$ the plane is divided into 2 regions

When $m = 2$ the plane can divided into as many as 4 regions

And, when $m = 3$ the plane can divided into as many as 7 regions



This data can also be collected into a table.

Lines	0	1	2	3	4	5	$\cdots$	m
Regions	1	2	4	7	11	16	$\cdots$	?

[1] What if you changed the problem to minimum number of regions?

Notice you could start at $m = 0$ and include the column

0
1

Can you write an explanation of this observation in English? Can you use induction in your explanation?

I've included columns for 4 and 5 lines, but left the general formula as a question.

Even more difficult to visualize is the maximum number of regions that 3 dimensional Euclidean space can be cut into by m flat planes. It turns out the sequence of regions starts out as 1, 2, 4, 8, 15, 26, ...

An $n-1$ dimensional space in an n dimensional space is called a hyperplane. That is, a point is a hyperplane on a line; a line is a hyperplane on a plane, a plane is a hyperplane in 3 dimensional space; and 3 dimensional space is a hyperplane in 4 dimensional space.

Although it may appear strange, it is useful to consider 0 dimensional space, a point, and how many regions it can be cut into, which of course is always 1. Collecting all of this data into one table allows a pattern to emerge.

Hyperplanes	0	1	2	3	4	5	$\cdots$	m
Regions in 0 Dimensions	1	1	1	1	1	1	$\cdots$	1
Regions in 1 Dimension	1	2	3	4	5	6	$\cdots$	$m+1$
Regions in 2 Dimensions	1	2	4	7	11	16	$\cdots$	?
Regions in 3 Dimensions	1	2	4	8	15	26	$\cdots$	?

It appears that if you add two consecutive numbers in a column the sum is the value in the next column, for instance, $7 + 8 = 15$ and $5 + 11 = 16$.

To express this observation a notation is needed for the values in the table. The depend on two variables: The dimension of the space being cut up and the number of hyperplanes. Let's agree to use n for the dimension of the space and m for the number of hyperplanes. Let's also agree to use the notation

$$\left\updownarrow \begin{matrix} n \\ m \end{matrix} \right\updownarrow$$

for the values in the table.

For instance

$$\left\updownarrow \begin{matrix} 2 \\ 4 \end{matrix} \right\updownarrow = 11$$

and

$$\left\updownarrow \begin{matrix} 3 \\ 5 \end{matrix} \right\updownarrow = 26$$

Using this notation the observation that sums of values in a column yields the value in the next column can be written as

$$\left\updownarrow \begin{matrix} n-1 \\ m-1 \end{matrix} \right\updownarrow + \left\updownarrow \begin{matrix} n \\ m-1 \end{matrix} \right\updownarrow = \left\updownarrow \begin{matrix} n \\ m \end{matrix} \right\updownarrow$$

This is called a recurrence equation. It computes values in the table using previously computed values.

Have you seen anything similar to this? Consider this table.

	0	1	2	3	4	5	$\cdots$	m
0	1	0	0	0	0	0	$\cdots$	0
1	1	1	0	0	0	0	$\cdots$	0
2	1	2	1	0	0	0	$\cdots$	0
3	1	3	3	1	0	0	$\cdots$	0
4	1	4	6	4	1	0	$\cdots$	0
5	1	5	10	10	5	1	$\cdots$	0

Use the notation $\binom{n}{m}$ for the value in row n, column m.

What is the recurrence equation for the above table?

What is an interpretation of the values?

Here are two other tables of useful numbers. Do you see the recurrence in them?

	0	1	2	3	4	5	$\cdots$	m
0	1	0	0	0	0	0	$\cdots$	0
1	0	1	0	0	0	0	$\cdots$	0
2	0	1	1	0	0	0	$\cdots$	0
3	0	2	3	1	0	0	$\cdots$	0
4	0	6	11	6	1	0	$\cdots$	0
5	0	24	50	35	10	1	$\cdots$	0

	0	1	2	3	4	5	$\cdots$	m
0	1	0	0	0	0	0	$\cdots$	0
1	0	1	0	0	0	0	$\cdots$	0
2	0	1	1	0	0	0	$\cdots$	0
3	0	1	3	1	0	0	$\cdots$	0
4	0	1	7	6	1	0	$\cdots$	0
5	0	1	15	25	10	1	$\cdots$	0

Use the notation $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ for the value in the first table and $\left\{\begin{smallmatrix} n \\ m \end{smallmatrix}\right\}$ for second table.

Let's return to the table that counts the maximum number of regions cut by hyperplanes and find functions for the values.

Hyperplanes	0	1	2	3	4	5	$\cdots$	m
Regions in 0 Dimensions	1	1	1	1	1	1	$\cdots$	1
Regions in 1 Dimension	1	2	3	4	5	6	$\cdots$	$m+1$
Regions in 2 Dimensions	1	2	4	7	11	16	$\cdots$	?
Regions in 3 Dimensions	1	2	4	8	15	26	$\cdots$	?

The function for values in the 0 dimensions row is a polynomial of degree 0, that is $p_0(m) = 1$. And, the function for values in the 1 dimension row is a polynomial of degree 1, that is $p_1(m) = m + 1$. It is reasonable to conjecture that the function for values in the 2 dimensions row is a polynomial of degree 2, that is $p_2(m) = am^2 + bm + c$ for some coefficients a, b and c. We can compute these values by computing differences and then using Newton's finite difference formula.

Please check that $p_2(m) = 1 + m(m+1)/2$ correctly computes the values in the table.

$p_2(m)$	$\triangle$	$\triangle^2$	$\triangle^3$
1			
	1		
2		1	
	2		0
4		1	
	3		0
7		1	
	4		0
11		1	
	5		
16			

$$\begin{aligned} p_2(m) &= p_2(0) + \triangle p_2(0)m + \frac{\triangle p_2(0)}{2} m(m-1) \\ &= 1 + m + \frac{m(m-1)}{2} \\ &= 1 + \frac{m(m+1)}{2} \end{aligned}$$

In the same manner the formula for $p_3(m)$ can be found.

Please check that $p_3(m) = 1 + m + m(m^2-1)/6$ correctly computes the values in the table.

$p_2(m)$	$\triangle$	$\triangle^2$	$\triangle^3$	$\triangle^4$
1				
	1			
2		1		
	2		1	
4		2		0
	4		1	
8		3		0
	7		1	
15		4		
	11			
26				

$$\begin{aligned} p_3(m) &= p_3(0) + \triangle p_3(0)m + \frac{\triangle p_3(0)}{2} m(m-1) + \frac{\triangle p_3(0)}{6} m(m-1)(m-2) \\ &= 1 + m + \frac{m(m-1)}{2} + \frac{m(m-1)(m-2)}{6} \\ &= 1 + m + \frac{m(m^2-1)}{6} \end{aligned}$$

Consider Pascal's triangle

	0	1	2	3	4	5	⋯	m
0	1	0	0	0	0	0	⋯	0
1	1	1	0	0	0	0	⋯	0
2	1	2	1	0	0	0	⋯	0
3	1	3	3	1	0	0	⋯	0
4	1	4	6	4	1	0	⋯	0
5	1	5	10	10	5	1	⋯	0

And a row of it, say 1, 3, 3, 1, 0^* What happens when you construct a differrence table for it?

The Kleene star notation 0^* means a repeated, unending sequence of zeros.

$\binom{3}{m}$	$\triangle$	$\triangle^2$	$\triangle^3$	$\triangle^4$
1				
	2			
3				
	0			
3				
	-2			
1				
	-1			
0				
	0			
0				

Problems on Recurrences

1. Find a recurrence equation and initial condition that enumerates the sequence of even natural numbers.
2. What change(s) must be made to your answer to problem 1 to enumerate the sequence of odd natural numbers?
3. Use mathematical induction to prove the following recurrence equations and initial conditions are solved by the given functions.

   (a) $t_n = 3t_{n-1}$, $t_0 = 2$, $t(n) = 2 \cdot 3^n$.
   (b) $t_n = 3t_{n-1} + 2$, $t_0 = 0$, $t(n) = 3^n - 1$.
   (c) $t_n = 5t_{n-1} - 6t_{n-2}$, $t_0 = 2$, $t_1 = 5$, $t(n) = 2^n + 3^n$.
   (d) $t_n = 2t_{n-1} + t_{n-2}$, $t_0 = 2$, $t_1 = -2$, $t(n) = (-1+\sqrt{2})^n + (-1-\sqrt{2})^n$.

4. Let $\vec{A} = \langle a_0, a_1, a_2, \ldots, a_n, \ldots \rangle$ be an arithmetic sequence sequence. That is, terms in $\vec{A}$ are given by a linear function $a(n) = mn + b$ for some slope m and y-intercept b. Find a recurrence equation and initial condition that enumerates the terms in the sequence.
5. Let $\vec{G} = \langle g_0, g_1, g_2, \ldots, g_n, \ldots \rangle$ be a geometric sequence sequence. That is, terms in $\vec{G}$ are given by a exponential function $g(n) = br^n$ for some ratio r and y-intercept b. Find a recurrence equation and initial condition that enumerates the terms in the sequence.

6. Use mathematical induction to prove that the common logarithm function $T_n = \log(n)$ satisfies the recurrence equation $T_{10n} = T_n + 1$ with initial condition $T_1 = 0$.
7. Use mathematical induction to prove that the function $r(n) = 2^{2^n} + 1$ solves the Fermat recurrence

$$r_n = r_{n-1} + 2^{2^{n-1}} r_0 \cdots r_{n-2}$$

with initial condition $r_0 = 3$.

8. Use mathematical induction to prove that the function $r(n) = 2^{2^n} + 1$ solves the Fermat recurrence

$$r_n = r_{n-1}^2 - 2(r_{n-2} - 1)^2$$

with initial condition $r_0 = 3$.

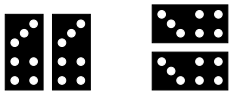
9. Use mathematical induction to prove that the function $r(n) = 2^{2^n} + 1$ solves the Fermat recurrence

$$r_n = r_0 \cdots r_{n-1} + 2$$

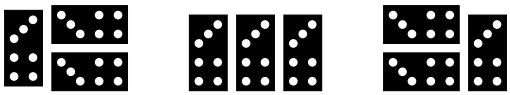
with initial condition $r_0 = 3$.

10. Morse code is a series of dots ($\cdot$) and dashes (—). Pretend a dot can be typed in 1 second while a dash takes 2 seconds to type. Find a recurrence equation that describes how many messages can be typed in n seconds.
11. In how many ways can $n \geq 1$ be expressed as a sum of 1's and 2's? Different term orders are considered different sums.
12. A domino is a rectangular tile 1 unit long and 2 units high

Dominos can be laid out to tile an $n \times 2$ area. For example, there are 2 ways to tile a 2×2 area



And 3 ways to tile a 3×2 area:



Find a recurrence relation that counts the number of $n \times 2$ tilings.

13. Pretend you invested \$1 on January 2, 2010. Pretend interest accrues at an annual compound rate of 10%. (For example, on January 2, 2011 and January 2, 2012 you will have \$1.10 and \$1.21 in your account.) How many years will you need to wait to save over \$1,000,000?
14. Pretend you invested \$1 on January 2 every year. Pretend interest accrues at an annual compound rate of 10%. (For example, on January 2, 2011 and January 2, 2012 you will have \$2.10 and \$1.00 + \$2.10 + \$0.21 = \$3.31 in your account) How many years will you need to wait to save over \$1,000,000?
15. Pretend you invested \$1 on the second day of every month. Pretend interest accrues at an monthly compound rate of 10%. How many months will you need to wait to save over \$1,000,000?

16. Consider the sequence generated by adding the alternating integers $1, -2, 3, -4, 5, \ldots$. The first few sums of these alternating integers are $\langle 1, -1, 2, -2, 3, \ldots \rangle$ Find a function $f(n)$ that generate these partial sums. That is, $f(0)$ should equal 1, $f(1)$ should equal -1, $f(2)$ should equal 2, $f(3)$ should equal -2, and so on.

17. Consider the strings of over the alphabet $\{a, b\}$ listed in order by length then alphabetically, that is,

$$\lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots$$

Name the terms in this sequence s_n so that $s_0 = \lambda$, $s_1 = a = s_0a$, $s_2 = s_0b$, $s_3 = s_1a = aa$, $s_4 = s_1b = ab$, and so on. Find a recursion equation that models the generation of these alphabetically ordered strings.