CSE 1400 Applied Discrete Mathematics
Relations

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Problems on Relations 18

Abstract

A relation $\sim$ describes how THINGS are connected. That a THING a is related to a THING b can be represented by

1. An ORDERED PAIR (a, b).
2. An directed edge $a\bullet \longrightarrow \bullet b$.
3. Or more commonly, simply using relational notation $a \sim b$.

A relation is

1. A set $\mathbb{G}$ of ORDERED PAIRS.
2. A directed graph G of nodes and edges.
3. A matrix of `True` and `False` values.

Higher-dimensional relations among a, b, c or more parameters can be defined. Higher-dimensional relations occur as tables in relational databases and as data in multi-variable problems.

Relations and Their Graphs

A RELATION IS A SET OF ORDERED PAIRS.

$$\mathbb{G} = \{(x, y) : x \sim y\} = \{(x, y) : x \text{ is related to } y.\}$$

There are many examples of relations. You are, no doubt, familiar with relations among people: Mother-Daughter, Father-Son, Parent-Child, Aunt-Nephew. Familial relations often become murky. We study relations that can be precisely defined. A few common relations are equality, congruence mod n, less than, divides, subset, and perpendicular.

In this course, relationships will be between two THINGS a and b. Relationships among 3 or more things are common and useful, but these ideas are not within the scope of this course.

A relation can be pictured of as a graph G.

$$\begin{tikzcd}
 & h\bullet \arrow[r] & \bullet g \arrow[d] \\
d\bullet & & c\bullet \arrow[ul] \\
 & \bullet e \arrow[dl] & \bullet f \arrow[l] \arrow[u] \\
\bullet a \arrow[r] & \bullet b \arrow[u] &
\end{tikzcd}$$

This graph represents the relation

$\mathbb{G} = \{(a, b), (b, c), (c, d), (d, h), (h, f), (f, e), (e, a)\}$

The course studies BINARY relations.

A Relation's Domain, Co-domain, and Range

THE THINGS INVOLVED IN A RELATION NEED NAMES: CALL THEM THE THINGS x AND y. Write

$$x \sim y$$

to express the phrase

"x IS RELATED TO y."

The value x belongs to a set $\mathbb{X}$ called the DOMAIN of $\sim$. The value y belongs to a set $\mathbb{Y}$ called the CO-DOMAIN of $\sim$.

The domain $\mathbb{X}$ is the set of elements that appear on the left-hand side of $\sim$. For this course, you can assume that every element in $\mathbb{X}$ appears on the left-hand side of $\sim$, that is, every relation we encounter is TOTAL, defined on all members of $\mathbb{X}$.

A relation $\sim$ is said to be TOTAL when every $x \in \mathbb{X}$ occurs on on the left-hand side of $\sim$ for some $y \in \mathbb{Y}$.

$$(\forall x \in \mathbb{X})(\exists y \in \mathbb{Y})(x \sim y)$$

If there is an $x \in \mathbb{X}$ such that no $y \in \mathbb{Y}$ is related to x, then $\sim$ is said to be PARTIAL.

$$(\exists x \in \mathbb{X})(\forall y \in \mathbb{Y})(x \not\sim y)$$

An instance of a partial relation on the real numbers is the pairing $(x, 1/x)$, which is undefined (has a pole) at $x = 0$.

The co-domain $\mathbb{Y}$ is the set of elements that can appear on the right-hand side of $\sim$. A potentially smaller set is the RANGE of $\sim$. The RANGE is the set of elements in $\mathbb{Y}$ that actually do appear on the right-hand side of $\sim$. In general, not every element y in $\mathbb{Y}$ occurs on the right-hand side of $\sim$. A relation is INTO its CO-DOMAIN and ONTO its RANGE. That is, the CO-DOMAIN of a relation, is the set of values y that could be related to some x. When a relation is onto its CO-DOMAIN every element in $\mathbb{Y}$ is related to some element x in $\mathbb{X}$, written

$$(\forall y \in \mathbb{Y})(\exists x \in \mathbb{X})(x \sim y)$$

A relation MAPS an element x in the DOMAIN $\mathbb{X}$ onto zero or more elements y in the RANGE $\mathbb{A} \subseteq \mathbb{Y}$.

A relation MAPS its domain INTO its co-domain and ONTO its range.

Consider the graph in figure 1 that depicts a relation from vertices x_0, x_1, x_2, x_3 in domain $\mathbb{X}$ to vertices y_0, y_1, y_2, y_3 in co-domain $\mathbb{Y}$

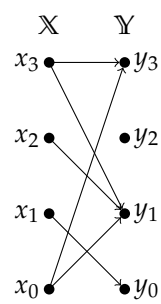


Figure 1: DOMAIN $\mathbb{X} = \{x_0, x_1, x_2, x_3\}$, CO-DOMAIN $\mathbb{Y} = \{y_0, y_1, y_2, y_3\}$, RANGE $\mathbb{A} = \{y_0, y_1, y_3\}$,

Here are some things to notice about figure 1.

1. The relation is total: there is at least one directed edge from each vertex in $\mathbb{X}$.

2. The relation is not onto its co-domain.

3. The relation is NON-DETERMINISTIC: A given input x can be related to many outputs y. For instance, in figure 1 x_0 is related to y_0 and y_1.

A relation is PARTIAL when a given input x cannot be related to any output y.

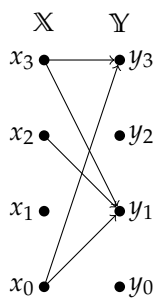


Figure 2: A partial relation: The relation is not defined on x_1.

A Sampling of Relations

You are familiar with many mathematical relations: Equality, less than, multiple of, and so on. These relations are between two THINGS : a and b, and are called BINARY relations.

Multidimensional relations are common in practice.

Equality

EQUALITY IS THE MOST BASIC RELATION.

$$5 = \frac{25}{5} = \frac{-10}{-2} = \frac{72}{\frac{72}{5}}$$

The name (5, 25/5, ...) can change, the THING remains the same.

```
equal :: Integer -> Integer -> Bool
equal a b = (0 == b - a)
```

Figure 3: Haskell for deciding equality on the integers.

Less than

Less than establishes an ORDER on the integers.

$$\cdots < -3 < -2 < -1 < 0 < 1 < 2 < 3 \cdots$$

```
lessthan :: Integer -> Integer -> Bool
lessthan a b = (0 < b - a)
```

Figure 4: Haskell for deciding less than on the integers.

Divides

A NATURAL NUMBER b DIVIDES A NATURAL NUMBER a WHEN THERE IS AN QUOTIENT $q \in \mathbb{N}$ SUCH $a = bq$. Divides is a relation on $\mathbb{N} \times \mathbb{N}$.

For instance, 15 divides 60 because

$$60 = 15 \cdot 4$$

Stated the other way around, 60 is a multiple of 15. On the other hand, 60 is not a multiple of 8; 8 does not divide 60. 60 divided by 8 leaves a remainder of 4. In general, let a and b be natural numbers. Then b divides a, if there is a natural number q such that $a = bq$.

7 divides 35. "b divides a" is written

$$b \mid a$$

"b does not divide a" is written

$$b \nmid a$$

"a is a multiple of b" is written

$$a \equiv 0 \mod b$$

"a is not a multiple of b" is written

$$a \not\equiv 0 \mod b$$

If b divides a, then a is a multiple of b.

The natural numbers can be partially ordered by the divides relation. The Hesse graph in figure 5 illustrates this ordering for the first few natural numbers.

Given natural numbers a and b, it is straightforward to write code to test if a divides b.

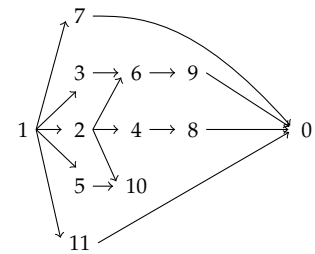


Figure 5: 1 is the smallest natural number with respect to divides; 1 divides every natural number. 0 is the largest natural number with respect to divides; every natural number divides 0.

```
divides :: Nat -> Nat -> Bool
divides a b = (a `div` b == 0)
```

Figure 6: Haskell for deciding divides on the natural numbers.
n is called the MODULUS.

Congruence Modulo n

CONGRUENCE MOD n HAS APPLICATIONS IN MANY AREAS, CRYPTOLOGY IS JUST ONE INTERESTING APPLICATION THAT CAN BE NAMED. When natural number n divides the difference $a - b$, the integers a and b are said to be congruent mod n. For instance, the following pairs are CONGRUENT mod 2.

$$(15, 9), \quad (26, 30), \quad (-9, 21), \quad (6, -28), \quad (17, -17)$$

while the following pairs are CONGRUENT mod 3.

$$(15, 9), \quad (26, 29), \quad (-8, 19), \quad (6, -27), \quad (17, -16)$$

Congruence collects pairs of integers based on their remainder when divided by n.

Mod 2 collects pairs (a, b) that are both even $(2s, 2t)$ or both odd $(2s+1, 2t+1)$. Congruence mod 2 partitions the integers into two EQUIVALENCE CLASSES: The even integers and the odd integers.

If both a and b are even, then their difference is divisible by 2. If both a and b are odd, then their difference is divisible by 2.

Mod 3 collects pairs (a, b) based on remainders upon division by 3. There are 3 cases.

1. Both a and b are multiples of three: $(3s, 3t)$
2. Both have a remainder of 1 when divided by three: $(3s+1, 3t+1)$
3. Both have a remainder of 2 when divided by three: $(3s+2, 3t+2)$

Congruence mod 3 partitions the integers into three EQUIVALENCE CLASSES: Integers that are multiples of 3, integers that are multiples of 3 plus 1, and integers that are multiples of 3 plus 2.

In general, two integers a and b are congruent mod n if $a - b$ is a multiple of n. Congruence mod n partitions the integers into n EQUIVALENCE CLASSES. A convenient notation for these sets of integers is

$$[0], [1], [2], \ldots, [n-1]$$

where

$$[r] = \{nk + r : k \in \mathbb{Z}\}$$

syntatic sugar can be added to denote the modulus: $[r]_n$.

For instance, mod 7, the set $[3]$ is

$$[3] = \{7s + 3 : s \in \mathbb{Z}\} = \{3, \pm 7 + 3, \pm 14 + 3, \ldots\}$$

while mod 11, the set [3] is

$$[3] = \{11s + 3 : s \in \mathbb{Z}\} = \{3, \pm 11 + 3, \pm 22 + 3, \ldots\}$$

Perpendicular on Lines

TWO LINES ARE PERPENDICULAR IF THEY INTERSECT AT RIGHT ANGLES. Recall a straight line can be written as a function

$$l(x) = mx + b = y$$

where l is the name of the function, the line, m is its slope, and b is its y-intercept, and x is an independent variable. Ordered pair (x, y) that satisfy the equation are related by lying on the line l.

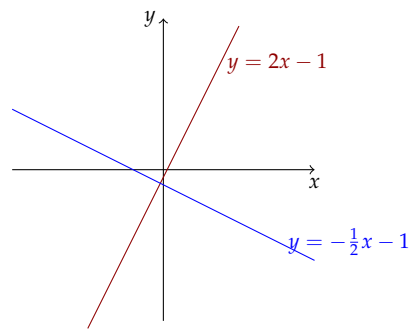
A straight line can be named by the ordered pair (m, b) of its slope and y intercept. Using analytic geometry it is simple to prove that two lines (m_0, b_0) and (m_1, b_1) are PERPENDICULAR when the product of their slopes is negative one

$$m_0 m_1 = -1$$

In which case we can write

$$(m_0, b_0) \perp (m_1, b_1).$$

When x takes on the values in the natural numbers, the linear function generates an arithmetic sequence.



The Incestuous and Empty Relations

The INCESTUOUS relation is where each element is related to every other element. A table of all 1's represents the incestuous relation. On the other hand, a table of all 0's represents the empty relation where no THING is related to any THING . The empty relation, being a set, is the empty set $\emptyset$.

Incestuous Relation

	a	*b*	*c*	*d*	*e*
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	1
e	1	1	1	1	1

A Relation is a Set of Ordered Pairs

A RELATION BETWEEN $\mathbb{X}$ AND $\mathbb{Y}$ IS A SUBSET OF THE CARTESIAN PRODUCT

$$\mathbb{X} \times \mathbb{Y} = \{(x, y) : x \in \mathbb{X} \wedge \mathbb{Y}\}$$

As instances consider the relations.

A relation is a set of ORDERED PAIRS.

The $=$ relation on the natural numbers the set of ordered pairs

$$\{(0, 0), (1, 1), (2, 2), \ldots\} = \{(n, n) : n \in \mathbb{N}\}$$

The less than relation ($<$) on the natural numbers the set of ordered pairs

$$\{(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3), \ldots\} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n < m\}$$

The divides relation ($|$) on the natural numbers the set of ordered pairs

$$\{(0, 0), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (1, 3), (3, 0), (3, 3), \ldots\} = \{(n, m) \in \mathbb{N} \times \mathbb{N} : n \mid m\}$$

Representing relations as ordered pair facilitate counting relations and relations with given properties. For instance, there are

$$2^{nm}$$

relations from $\mathbb{X}$ to $\mathbb{Y}$ when $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$.

There are nm elements (ordered pairs) in $\mathbb{X} \times \mathbb{Y}$. A set with nm elements has 2^{nm} subsets

The Inverse Relation

THE INVERSE OF A RELATION CHANGES THE ORDER OF A PAIR OF RELATED THINGS . Let $\mathbb{L}$ be a relation from $\mathbb{X}$ to $\mathbb{Y}$. Write $x \sim y$ when x is related to y, that is, when $(x, y) \in \mathbb{L}$.

$$\mathbb{L} = \{(x, y) : x \in \mathbb{X} \wedge y \in \mathbb{Y} \wedge x \sim y\}$$

Then the inverse of $\mathbb{L}$ is the set

$$\mathbb{L}^{-1} = \{(y, x) : x \in \mathbb{X} \wedge y \in \mathbb{Y} \wedge x \sim y\}$$

For instance, the inverse of the relation $<$ is the relation $>$ and the inverse of divides is multiple of.

Counting Relations

HOW MANY RELATIONS CAN BE DEFINED FROM $\mathbb{X}$ TO $\mathbb{Y}$? The answer depends on the sizes, the cardinalities, of the two sets. Pretend these sets have cardinality $n = |\mathbb{X}|$ and $m = |\mathbb{Y}|$. As a first step it is important to be clear about what we are to count. By definition, a relation from $\mathbb{X}$ to $\mathbb{Y}$ is a subset of the Cartesian product $\mathbb{X} \times \mathbb{Y}$. Answering the question REDUCES to counting subsets.

The cardinality of the Cartesian product is is the count of ordered pairs (x, y) where $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. This count is

$$|\mathbb{X} \times \mathbb{Y}| = |\mathbb{X}| \cdot |\mathbb{Y}| = nm$$

Every subset of $\mathbb{X} \times \mathbb{Y}$ is a relation; every relation is a subset of $\mathbb{X} \times \mathbb{Y}$. There are 2^{nm} subsets of an nm-element set.

Theorem 1 (Counting Relations)**.** *Let the cardinalities of set $\mathbb{X}$ and $\mathbb{Y}$ by $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$. There are 2^{nm} relations from $\mathbb{X}$ to $\mathbb{Y}$. In particular, there are 2^{n^2} relations on $\mathbb{X}$.*

Recall there are $\binom{nm}{k}$ k-element subsets of a set with nm-elements. The sum of $\binom{nm}{k}$ as k ranges from 0 to nm counts the number of subsets of $\mathbb{X} \times \mathbb{Y}$.

Recall from the binomial theorem that the sum of binomial coefficients in row nm of Pascal's triangle is 2^{nm}.

Relational Properties

There are properties of relations that allow things to be sorted or decide that two things are equivalent. A relation $\sim$ can have several properties. A relation $\sim$ could be reflexive, symmetric, antisymmetric, or transitive.

Reflexive Property

If every element $a \in \mathbb{A}$ is related to itself, then the relation is reflexive on $\mathbb{A}$.

$$(\forall a \in \mathbb{A})(a \sim a)$$

The following relations are reflexive.

1. Equality on the integers. Proof: $a = a$ for all $a \in \mathbb{Z}$.
2. Congruence mod n on the integers. Proof: For all $a \in \mathbb{Z}$, $a - a = 0$ is a multiple of n. Therefore $a \equiv a \bmod n$ for all integers a.
3. Divides on the natural numbers. Proof: For all $n \in \mathbb{N}$, $n = n \cdot 1$. Therefore $n \mid n$ for all natural numbers n.

To show that a node is related to itself, draw a "self-loop"  It is common not to draw self-loops for a reflexive relation; they clutter the graph. The exceptions are finite state machines where self-loops show a transition from a state directly back to itself.

The next list of relations are not reflexive.

1. Not equal on the set of natural numbers. Counterexample: $0 \neq 0$ is false.
2. Less than on the integers. Counterexample: $0 < 0$ is false.
3. Perpendicular on the set of lines. Counterexample: The line $x - y = 0$ is not perpendicular to itself.

When a relation is represented as a table you can see if it is reflexive or not. When the main diagonal, from upper-left to lower-right, contains only 1's the table states each element is related to itself. That is, the relation is reflexive.

Tables also provide a convenient tool to count reflexive relations on a set $\mathbb{A}$. To be reflexive, the values on the main diagonal must be 1, but the off-diagonal values can either 0 or 1. Therefore, there are

$$2^{(\text{count of off-diagonal elements})}$$

reflexive relations on a set. That is, there are

$$2^{n(n-1)}$$

A Reflexive Relation

	a	b	c	d	e
a	1	1	0	1	0
b	0	1	0	1	0
c	0	1	1	0	1
d	1	1	0	1	1
e	1	0	1	1	1

reflexive relations on a set with cardinality n. The problem of counting reflexive relations REDUCES to counting off-diagonal elements in an $n \times n$ table.

Pretend the cardinality of $\mathbb{A}$ is $|\mathbb{A}| = n$. Then there are n^2 elements in the table and n of them lie along the main diagonal. Therefore, there are $n^2 - n$ off-diagonal elements.

$n^2 - n = n(n-1)$

You can also count the number of off-diagonal elements directly. For instance, count there are 1, 2, 3 up to $n-1$ elements along diagonals in the lower triangle of the table. The sum of natural numbers from 0 to $n-1$ is a triangular number.

$$\sum_{0 \le k < n} k = 0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

Double this count to include the off-diagonal elements above that main diagonal too. Conclude that there are $n(n-1)$ off-diagonal elements. Knowing the number of off-diagonal elements, compute that there are $2^{n(n-1)}$ reflexive relations on an n-element set.

Symmetric Property

A RELATION IS SYMMETRIC IF IT IS ITS OWN INVERSE. When a is related to b, b must also be related to a.

$$(\forall a \in \mathbb{A})(\forall b \in \mathbb{A})((a \sim b) \to (b \sim a))$$

The graph of a symmetric relation is undirected. The edge between related nodes x and y does not need arrowhead.

$\bullet a$ —— $\bullet b$

A precisionist might draw arrowheads at both ends. $\bullet a \longleftrightarrow \bullet b$

Symmetry is fundamental to defining equivalent (not equal) THINGS .

The following relations are symmetric.

1. Equality on the integers. proof: If $a = b$, then $b = a$ for all $a, b \in \mathbb{Z}$.
2. Inequality on the integers. proof: If $a \neq b$, then $b \neq a$ for all $a, b \in \mathbb{Z}$.
3. Congruence mod n on the integers. proof: If n divides $a - b$, then n divides $b - a$.
4. Parallel on the set of lines in the two dimensional Cartesian plane. proof: If l is parallel to l, then l is parallel to l.
5. Perpendicular on the set of lines in the two dimensional Cartesian plane. proof: If l is perpendicular to λ, then λ is perpendicular to l.

The next list of relations are NOT symmetric.

1. Less than on the integers. counterexample: 2 is less than 3, but 3 is not less than 2.

2. Divides on the natural numbers. counterexample: 2 divides 4, yet 4 does not divide 2.

3. Subset on the power set of a set. counterexample: $\{0\}$ is a subset of $\mathbb{B} = \{0, 1\}$, but $\{0, 1\}$ is not a subset of $\{0\}$.

When a relation is represented as a table you can to see if it is symmetric or not. Symmetry is a property between elements in the upper and lower triangle of a matrix.

When the matrix is folded along the diagonal, the values in the upper and lower triangle align. The value in row i, column j equals the value in row j, column i. When 1 is in row i, column j, then 1 is in the symmetric location: row j, column i. When 0 is in row i, column j, then 0 is in the symmetric location: row j, column i.

Tables also provide a convenient tool to count symmetric relations on a set $\mathbb{A}$. To compute this count consider the number of main diagonal and off-diagonal elements in the table. To be symmetric, the values on the main diagonal can be either 0 or 1. However, off-diagonal values are constrained by the rule:

The value in row i, column j matches the value in row j, column i.

That is, choosing a value for a lower triangular element determines the value for its symmetric upper triangular element. Since the triangular number $n(n-1)/2$ counts the elements in the lower triangle, there are $2^{n(n-1)/2}$ ways to set the off-diagonal elements in a symmetric table. Therefore, there are

$$2^{(\text{count of main diagonal elements})}2^{(\text{count of off-diagonal elements})/2}$$

symmetric relations on a set with cardinality n. That is, there are

$$2^n 2^{n(n-1)/2} = 2^{n(n+1)/2} = \sqrt{2^{n(n+1)}}$$

symmetric relations on a set with cardinality n.

A Symmetric Relation

	a	b	c	d	e
a	1	1	0	1	0
b	1	1	0	1	0
c	0	0	1	0	1
d	1	1	0	1	1
e	0	0	1	1	1

Notice when you fold the matrix along its main diagonal the values align.

Counts of reflexive reflexive relations was computed in a similar manner above.

There are 2^n different ways to set the value of elements on the main diagonal.

Antisymmetric Property

A RELATION IS ANTISYMMETRIC WHEN a IS RELATED TO b AND b IS RELATED TO a ONLY IF a AND b ARE EQUAL.

Antisymmetry is fundamental deciding how to order THINGS .

$$(\forall a)(\forall b)((a \sim b \wedge b \sim a) \rightarrow (a = b))$$

$$(\forall a)(\forall b)((a \not\sim b \vee b \not\sim a) \vee (a = b))$$

Use the logical equivalence $(P \rightarrow Q) \equiv (\neg P \vee Q)$ and one of DeMorgan's laws to write antisymmetry as a disjunction.

The following relations are antisymmetric.

1. Equality on the integers. proof: For all $a, b \in \mathbb{Z}$, "if $a = b$ and $b = a$, then $a = b$" is `True`.
2. Less than on the integers. proof: For all $a, b \in \mathbb{Z}$, "if $a < b$ and $b < a$, then $a = b$" is `True`.
3. Divides from the natural numbers to the integers. proof: if a divides b and b divides a, then, by the domain definition for divides, both a and b are natural numbers. Also there are natural numbers c and d such that $ac = b$ and $bd = a$. Therefore $acd = bd = a$, $cd = 1$, and $c = d = 1$.
4. Subset on the power set of a set.

That equality is both symmetric and antisymmetric shows antisymmetry is not the same as not symmetric.

The next list of relations are not antisymmetric.

1. Inequality on the integers. counterexample: "if $0 \neq 1$ and $1 \neq 0$, then $0 = 1$." is `False`.
2. Congruence mod n on the integers.
3. Parallel on the set of lines.
4. Perpendicular on the set of lines.

An Antisymmetric Relation

	a	b	c	d	e
a	1	1	0	0	0
b	0	1	0	1	0
c	0	0	1	0	1
d	1	0	0	1	0
e	0	0	1	1	1

Notice when you fold the matrix along its main diagonal the off-diagonal 1's align with 0's.

When a relation is represented as a table you can see if it is antisymmetric or not. When 1 occurs in row i, column j, then a 0 must appear in the symmetric location: row j, column i.

Transitive Property

A RELATION IS TRANSITIVE WHEN a IS RELATED TO b AND b IS RELATED TO c IMPLIES a IS RELATED TO c. The following relations are transitive.

Transitivity is key to keeping relations consistent.

1. Equality on the integers. proof: For all $a, b \in \mathbb{Z}$, "if $a = b$ and $b = a$, then $a = b$" is `True`.
2. Less than on the integers. proof: For all $a, b \in \mathbb{Z}$, "if $a < b$ and $b < a$, then $a = b$" is `True`.
3. Congruence mod n on the integers.

4. Divides from the natural numbers to the integers. proof: if a divides b and b divides a, then, by the domain definition for divides, both a and b are natural numbers. Also there are natural numbers c and d such that $ac = b$ and $bd = a$. Therefore $acd = bd = a$, $cd = 1$, and $c = d = 1$.

5. Subset on the power set of a set.

6. Parallel on the set of lines.

The next list of relations are not transitive.

1. Inequality on the integers. counterexample: "if $0 \neq 1$ and $1 \neq 0$, then $0 \neq 0$." is `False`.

2. Perpendicular on the set of lines.

3. Before in three dimensional space. Consider a pinwheel an Escher drawing(?)

Orders and Equivalences

TWO IMPORTANT CLASSES OF RELATIONS are ORDERS and EQUIVALENCES. Orders describe how one THING is before or after another. Equivalences allow grouping of things based on common attributes.

Orders

AN ORDER RELATION ESTABLISHES A PRECEDENCE between THING S: You can decide if one THING is before or after another THING . TOTAL ORDERS allow before or after comparison between every pair of THING S. For instance, $\leq$ on the integers is a total order. An order is PARTIALS when only some pairs of THING S can be placed in a precedence graph. For instance, the subset relation on the power set of a set is a partial order.

A PARTIAL ORDER is a relation $\sim$ that is reflexive, antisymmetric, and transitive. The antisymmetric is The following relations are partial orders.

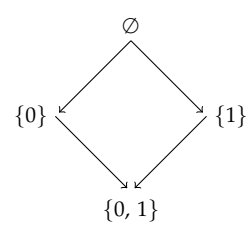


Figure 7: The subset precedence graph on the power set of $\{0, 1\}$. The subsets $\{0\}$ and $\{1\}$ are not related.

1. Equality on the integers: $a = b$ for $a, b \in \mathbb{Z}$.

2. Less than or equal on the integers: $a \leq b$ for $a, b \in \mathbb{Z}$.

3. Divides on the natural numbers: $a \mid b$ for $a, b \in \mathbb{N}$.

4. Subset on the power set $2^{\mathbb{A}}$ of a set $\mathbb{A}$: $\mathbb{X} \subseteq \mathbb{Y}$ for $\mathbb{X}, \mathbb{Y} \subseteq \mathbb{A}$.

The following relations are not partial orders.

1. Inequality on the integers.
   (a) Inequality is not reflexive: $a \neq a$ is `False` for every integer $a \in \mathbb{Z}$.
   (b) Inequality is not antisymmetric: if $a \neq b$ and $b \neq a$, then $a = b$ is `False`.
   (c) Inequality is not transitive: if $a \neq b$ and $b \neq c$, then $a \neq c$ can be `False`. In particular, transitivity fails when $a = c$.
2. Congruence mod n on the integers
   (a) Congruence mod n is not antisymmetric.
   A counterexample to the statement congruence mod n is antisymmetric is $2 \equiv 0 \bmod 2$ but $2 \neq 0$.
   Consider the general case. That is, when integers a and b have the identical remainders when divided by n, they are congruence, but not necessarily equal. That is, when integers $a = nc + r$ and $b = nd + r$, and $c \neq d$, then $a \equiv b \bmod n$ and $b \equiv a \bmod n$, but $a \neq b$
3. Parallel on the set of lines in two dimensions.
   (a) Parallel is not antisymmetric: Two lines can be parallel without being equal.
4. Perpendicular on the set of lines: Two lines can be perpendicular without being equal.

The integers mod n can be strictly ordered by the inherited less than relation on the integers: $0 < 1 < 2 < \cdots (n-1)$. However, this is not the congruence mod n relation.

The natural order on an alphabet can be extended to strings over the alphabet.

For instance, the English alphabet is naturally ordered

$$a < b < c < \cdots < x < y < z$$

Definition 1 (Alphabetic Order)**.** 1. The empty string λ precedes every string s.

2. If s and t are strings of length 1 or greater, then $s = \alpha u$ and $t = \beta v$ for some characters $\alpha, \beta \in \mathbb{A}$ and

$$s \leq t = \begin{cases} \texttt{True} & \text{if } \alpha < \beta \\ \texttt{True} & \text{if } \alpha = \beta \text{ and } u \leq v \\ \texttt{False} & \text{otherwise} \end{cases}$$

Well-Ordered Sets

A SET IS WELL-ORDERED WHEN EVERY NON-EMPTY SUBSET HAS A LEAST ELEMENT. A useful theorem is that any non-empty subset $\mathbb{A}$ of natural numbers has a least element under the standard $\leq$ relation.

Theorem 2 (Well-Ordering)**.** *Let $\mathbb{A}$ be a non-empty subset of the natural numbers. There exists an element $a \in \mathbb{A}$ such that $a \leq b$ for all $b \in \mathbb{A}$. This element a is called the least element of $\mathbb{A}$.*

Writing the statement in predicate form yields

$$(\exists a \mathbb{A})(\forall b \in \mathbb{A})(a \leq b)$$

which helps to show the absurdity of its negation.

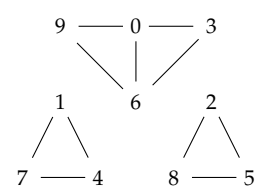
$$\neg(\exists a \in \mathbb{A})(\forall b \in \mathbb{A})(a \leq b) = (\forall a \in \mathbb{A})(\exists b \in \mathbb{A})(b > a)$$

The well-ordering principle for the natural numbers is equivalent to Peano's axiom of induction.

Equivalences

An Equivalence relation partitions a set into a collection of disjoint subsets called equivalence classes. The adjacency matrix for congruence mod 3 on the digits is shown below. The equivalence classes can be better seen if you permute the rows and columns.

For instance, congruence mod 3 partitions the digits into 3 equivalence classes.



Congruence mod 3 on the digits

	0	1	2	3	4	5	6	7	8	9
0	1	0	0	1	0	0	1	0	0	1
1	0	1	0	0	1	0	0	1	0	0
2	0	0	1	0	0	1	0	0	1	0
3	1	0	0	1	0	0	1	0	0	1
4	0	1	0	0	1	0	0	1	0	0
5	0	0	1	0	0	1	0	0	1	0
6	1	0	0	1	0	0	1	0	0	1
7	0	1	0	0	1	0	0	1	0	0
8	0	0	1	0	0	1	0	0	1	0
9	1	0	0	1	0	0	1	0	0	1

Relation	Pair	Graph	Matrix	Partition
$0 \sim 0$	$\{(0, 0)\}$	$0\bullet$	$\begin{array}{cc} & 0 \\ \hline 0 & 1 \end{array}$	$\{\{0\}\}$

Figure 8: Describing an equivalence relation on $\{0\}$.

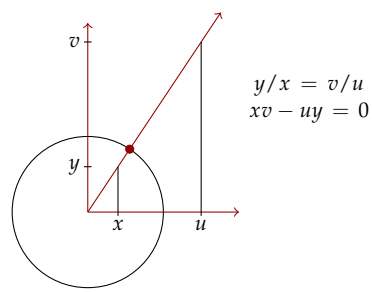
Congruence mod 3 on the digits

	0	3	6	9	1	4	7	2	5	8
0	1	1	1	1	0	0	0	0	0	0
3	1	1	1	1	0	0	0	0	0	0
6	1	1	1	1	0	0	0	0	0	0
9	1	1	1	1	0	0	0	0	0	0
1	0	0	0	0	1	1	1	0	0	0
4	0	0	0	0	1	1	1	0	0	0
7	0	0	0	0	1	1	1	0	0	0
2	0	0	0	0	0	0	0	1	1	1
5	0	0	0	0	0	0	0	1	1	1
8	0	0	0	0	0	0	0	1	1	1

Computer graphics depends on HOMOGENEOUS COORDINATES. In two-dimensions, the homogeneous coordinates (x, y) and (u, v) are EQUIVALENT if

$$xv - uy = 0$$

That is, two homogeneous points (x, y) and (u, v) are equivalent if they lie on the same ray through the origin. The standard name for a homogeneous point is the one lying on the unit circle $x^2 + y^2 = 1$. Homogeneous coordinates establish an equivalence relation on the punctured plane $\mathbb{R} - \{(0, 0)\}$.



Homogeneous coordinates PROJECT point onto the unit circle.

Do you see why the origin was left out? It is convenient to define the $\emptyset$ as an equivalence relation on itself, so there is 1. Otherwise, the relation is not an equivalence relation on any other set.

[1] Self-loops are not drawn.

Equivalence Relations Partition a Set

FOR SMALL SETS EACH EQUIVALENCE RELATION CAN BE LISTED. Let $\mathbb{A}$ be a set with cardinality $|\mathbb{A}| = n$. There are many notations for describing a relation: Itself, a set of ordered pairs, a graph [1], or a matrix

1. When $\mathbb{A} = \{0\}$ and $n = 1$ there is one equivalence relation. It is described in figure 8.
2. When $\mathbb{A} = \{0, 1\}$ and $n = 2$ there are two different equivalence relations that can be defined. They are described in figure 9.
3. When $\mathbb{A} = \{0, 1, 2\}$ and $n = 3$ there are 5 different equivalence relations that can be defined. Many of the notations are cumbersome. Listing partitions of the set seems less cumbersome. In the

diagram below partitions of $\mathbb{A}$ are listed based on the number of subsets in the partition: 1, 2, or 3.

Partitions		
$\{\{0, 1, 2\}\}$	$\{\{0\}, \{1, 2\}\}$	$\{\{0\}, \{1\}, \{2\}\}$
	$\{\{1\}, \{0, 2\}\}$	
	$\{\{2\}, \{0, 1\}\}$	

4. When $\mathbb{A} = \{0, 1, 2, 3\}$ and $n = 4$ there are 15 different equivalence relations that can be defined.

Partitions			
$\{\{0, 1, 2, 3\}\}$	$\{\{3\}, \{0, 1, 2\}\}$	$\{\{0\}, \{1, 2\}, \{3\}\}$	$\{\{0\}, \{1\}, \{2\}, \{3\}\}$
	$\{\{0\}, \{1, 2, 3\}\}$	$\{\{1\}, \{0, 2\}, \{3\}\}$	
	$\{\{1\}, \{0, 2, 3\}\}$	$\{\{2\}, \{0, 1\}, \{3\}\}$	
	$\{\{2\}, \{0, 1,3\}\}$	$\{\{0\}, \{1\}, \{2, 3\}\}$	
	$\{\{0, 3\}, \{1, 2\}\}$	$\{\{0\}, \{1, 3\}, \{2\}\}$	
	$\{\{1, 3\}, \{0, 2\}\}$	$\{\{0, 3\}, \{1\}, \{2\}\}$	
	$\{\{2, 3\}, \{0, 1\}\}$		

If you study the partitions, you'll find an algorithm that computes the number of partitions with a given number of subsets.

Consider the seven 2-subset partitions of $\{0, 1, 2, 3\}$. They were constructed from the 1-subset and 2-subset partitions of $\{0, 1, 2\}$.

1. The union of $\{\{3\}\}$ with the 1-subset partition $\{\{0, 1, 2\}\}$ formed a 2-subset partition $\{\{0, 1, 2\}, \{3\}\}$ of $\{0, 1, 2, 3\}$.

2. The union of $\{3\}$ with each of the two sets in the three 2-subset partition of $\{0, 1, 2\}$ formed six more 2-subset partitions of $\{0, 1, 2, 3\}$.

This construction is general.

For instance, the six 3-subset partitions of $\{0, 1, 2, 3\}$, were constructed from the 2-subset and 3-subset partitions of $\{0, 1, 2\}$.

Stirling Numbers of the Second Kind

A SET WITH CARDINALITY n CAN BE PARTITIONED INTO m SUBSETS IN $\left\{ {n \atop m} \right\}$ WAYS. Stirling numbers of the second kind are defined by the recurrence equation

$$\left\{ {n \atop m} \right\} = \left\{ {n-1 \atop m-1} \right\} + m \left\{ {n-1 \atop m} \right\}$$

with boundary conditions

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = 0 \quad \text{and} \quad \left\{ {n \atop n} \right\} = 1, \text{ for } n > 0$$

Check that the following arithmetic can be verified by the numbers in table 1.

$$\left\{ {4 \atop 3} \right\} = \left\{ {3 \atop 2} \right\} + 3 \left\{ {3 \atop 3} \right\} = 3 + 3 \cdot 1$$
$$\left\{ {5 \atop 3} \right\} = \left\{ {4 \atop 2} \right\} + 3 \left\{ {4 \atop 3} \right\} = 7 + 3 \cdot 6$$
$$\left\{ {7 \atop 5} \right\} = \left\{ {6 \atop 4} \right\} + 5 \left\{ {6 \atop 5} \right\} = 65 + 5 \cdot 15$$

The notation $\left\{ {n \atop m} \right\}$ is called *n subset m*.

Table 1: Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$ count the number of partitions of n THINGS into m subsets.

	Stirling Numbers of the Second Kind $\left\{ {n \atop m} \right\}$								
	Subset m								
	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0
3	0	1	3	1	0	0	0	0	0
n 4	0	1	7	6	1	0	0	0	0
5	0	1	15	25	10	1	0	0	0
6	0	1	31	90	65	15	1	0	0
7	0	1	63	301	350	140	21	1	0
8	0	1	127	966	1701	1050	266	28	1

Problems on Relations

1. Describe the relation $P \sim Q = \texttt{True}$ over the set of Boolean variables $P, Q \in \mathbb{B}$ when $\sim$ is each of the Boolean operators $\wedge$, $\vee$, $\rightarrow$, $\equiv$.
2. How many relations can be defined on the following sets?
   (a) The sets of bits $\mathbb{B} = \{0, 1\}$.
   (b) The sets of digits $\mathbb{D} = \{0, 1, \ldots, 9\}$.
   (c) From the bits to the digits.
   (d) From $\mathbb{B}^n$, the bit strings of length n, to the bits $\mathbb{B}$.

3. A relation on an n-element set $\mathbb{A}$ can be represented by an $n \times n$ adjacency matrix. Given that there are

$$2^{(\text{count of off-diagonal elements})}$$

reflexive relations on a set, how many reflexive relations are there for $n = 0, 1, 2, 3$ and 4?

4. Given that there are

$$f(n) = 2^n 2^{n(n-1)/2} = 2^{n(n+1)/2} = \sqrt{2^{n(n+1)}}$$

symmetric relations on a set with cardinality n. Compute the value of $f(n)$ for $n = 0, 1, 2, 3$ and 4.

5. Describe what must be true for two lines, written as slope-intercept pairs (m_0, b_0) and (m_1, b_1), to be perpendicular. Use the notation

$$(m_0, b_0) \perp (m_1, b_1)$$

The line (m, b) is described by the equation $y = mx + b$.

to denote the perpendicular relationship between two lines.

6. Describe what must be true for two lines, written in standard form (a_0, b_0, c_0) and (a_1, b_1, c_1), to be perpendicular. Use the notation

$$(a_0, b_0, c_0) \perp (a_1, b_1, c_1)$$

The line (a, b, c) is described by the equation $ax + by = c$.

to denote the perpendicular relationship between two lines.

7. Describe what must be true for two lines, written as slope-intercept pairs (m_0, b_0) and (m_1, b_1), to be parallel. Use the notation

$$(m_0, b_0) \| (m_1, b_1)$$

to denote the parallel relationship between two lines.

8. Interpret the following ordered pairs as slope-intercept descriptions of lines. Which pairs are perpendicular? Which pairs are parallel?

   (a) $(1, 2)$ and $(-1, 3)$.
   (b) $(3, 2)$ and $(3, 3)$.
   (c) $(-0.5, 2)$ and $(2, 3)$.
   (d) $(-3, 2)$ and $(4, 3)$.

9. For each relation, determine if it is reflexive, symmetric, antisymmetric, or transitive.

   (a) Equality on the natural numbers, written $a = b$.
   (b) Divides on the natural numbers, written $a \mid b$.
   (c) Subset on the power set $2^{\mathbb{A}}$ of $\mathbb{A}$, written $\mathbb{X} \subseteq \mathbb{Y}$ for subsets $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbb{A}$.
   (d) Congruence mod m on the integers, written $a \equiv b \bmod m$.
   (e) $x \sim y$ if $x - y = 9$.
   (f) $x \sim y$ if $|x - y|$ is prime.

10. Let $r = (a, b)$ and $s = (c, d)$ be ordered pairs of integers. (You could think of r and s as rational numbers.) If the cross-product ad and cb are equal say r is equivalent to s and write $r \equiv s$.

   (a) Give instances of $r = (a, b)$ and $s = (c, d)$ that are equivalent.
   (b) Give instances of $r = (a, b)$ and $s = (c, d)$ that are not equivalent.
   (c) What pairs are equivalent to the pair $(1, 1)$?
   (d) Is is $r \equiv s$ reflexive? Explain.
   (e) Is $r \equiv s$ symmetric? Explain.
   (f) Is $r \equiv s$ transitive? Explain.

(g) Is $r \equiv s$ an equivalence relation? Explain.

(h) How does this relation describe equivalent names for rational number fractions?

(i) How does this relation describe a geometric relationship between points in a two-dimensional space?

11. Let a, b, c, d be integers and say the ordered pair (a, b) related to (c, d) if $a + d = b + c$.

$$(a, b) \sim (c, d) \quad \text{if } a + d = b + c$$

(a) Is $\sim$ reflexive? Explain.

(b) Is $\sim$ symmetric? Explain.

(c) Is $\sim$ transitive? Explain.

(d) Is $\sim$ an equivalence relation? Explain.

12. Say that two floating point numbers x and y are approximately equal if the differ by $\epsilon = 1/32 = 3.125 \times 10^{-2}$.

$$x \approx y \quad \text{if and only if } |x - y| \leq \epsilon$$

(a) Is approximately equal reflexive? Explain.

(b) Is approximately equal symmetric? Explain.

(c) Is approximately equal antisymmetric? Explain.

(d) Is approximately equal transitive? Explain.

(e) Is *approximately equal* an equivalence relation? Explain.

(f) Is *approximately equal* a partial order? Explain.

13. On the set of real numbers $\mathbb{R} = \{x : -\infty < x < \infty\}$ define the relation

$$\mathbb{S} = \{(x, y) : x, y \in \mathbb{R}, \text{ and } x - y \text{ is an integer}\}$$

Show that $\mathbb{S}$ is an equivalence relation on $\mathbb{R}$.

14. Consider the childhood game "rock beats scissors, scissors beat paper, paper beats rock."

(a) Construct the adjacency matrix that represents the "beats" relation.

(b) Is "beats" reflexive, symmetric, antisymmetric, transitive?

15. Let $\vec{R}$ and $\vec{S}$ be sequences of natural numbers. Say that sequence $\vec{R}$ precedes $\vec{S}$ and write $\vec{R} \leq \vec{S}$ if $r_i \leq s_i$ for all $i = 0, 1, 2, \ldots$ As an example, $\vec{R} = \langle 0, 1, 2, 3, \ldots \rangle$ precedes $\vec{S} = \langle 1, 2, 4, 8, \ldots \rangle$ because $i \leq 2^i$ for all natural numbers i.

(a) Show that precedes is a partial order.

(b) Draw a graph showing the order of the 8 finite sequences

$$\begin{array}{cccc} \langle 0, 0, 0 \rangle & \langle 0, 0, 1 \rangle & \langle 0, 1, 0 \rangle & \langle 0, 1, 1 \rangle \\ \langle 1, 0, 0 \rangle & \langle 1, 0, 1 \rangle & \langle 1, 1, 0 \rangle & \langle 1, 1, 1 \rangle \end{array}$$

16. The U. S. Post Office can increase its efficiency by sorting letters based on width and height. Define a partial order on a set of ordered pairs (w, h) that describe the width and height of a letter.

17. The U. S. Post Office also ships packages that have a width, height, and depth. Define a partial order on ordered triples (w, h, d) to help the U.S.P.O. sort packages.

18. Negate the definition of "relation $\sim$ is reflexive:"

$$(\forall x \in \mathbb{X})(x \sim x)$$

to define "relation $\sim$ is not reflexive."

19. Negate the definition of "relation $\sim$ is symmetric:"

$$(\forall x \in \mathbb{X})(\forall y \in \mathbb{X})((x \sim y) \implies (y \sim x))$$

to define "relation $\sim$ is not symmetric."

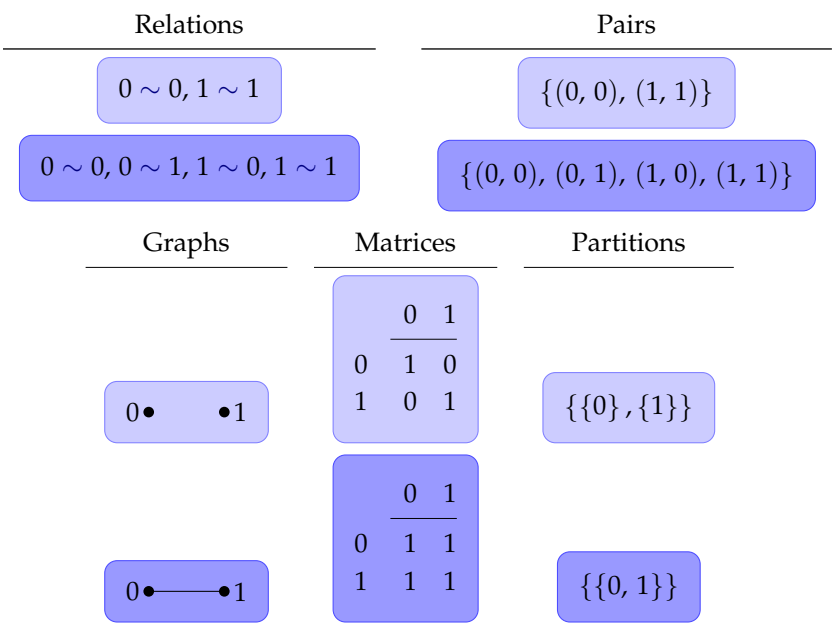


Figure 9: Describing two equivalence relations on $\{0, 1\}$.