Name:

CSE 1400 Applied Discrete Mathematics

Spring 2015 Week 6 Key

1. *For all* ($\forall$) and *there exists* ($\exists$) are important quantifiers. $\forall$ allows you to say a predicate statement is always `True` or always `False`.

$$(\forall\ x)(\mathrm{P}(x)) \quad \text{or} \quad (\forall\ x)(\neg\mathrm{P}(x))$$

$\exists$ allows you to say a predicate statement is `True` or `False` for one or more values

$$(\exists\ x)(\mathrm{P}(x)) \quad \text{or} \quad (\exists\ x)(\neg\mathrm{P}(x))$$

These quantifiers play together in strange and mysterious ways. It is worthwhile to learn the basics.

Consider the predicates below. Put $(\forall\ x)$ in front of each of them and decide if the quantified statements is `True` or `False`. Do the same for $(\exists\ x)$. I'll use the notation :: to indicate how a predicate is defined. Assume the domain is the real numbers $\mathbb{R}$.

(a) $\mathrm{P}(x) :: x = x + 1$
Answer: Neither $(\forall\ x)(x = x + 1)$ nor $(\exists x)(x = x + 1)$ is `True`.

(b) $\mathrm{P}(x) :: x^2 - x - 1 = 0$
Answer: $(\forall\ x)(x^2 - x - 1 = 0)$ is `False`, but $(\exists x)(x^2 - x - 1 = 0)$ is `True`. The roots of this polynomial equation are $\varphi = (1 + \sqrt{5})/2$ and $\bar{\varphi} = (1 - \sqrt{5})/2$, the golden ratio and its conjugate.

(c) $\mathrm{P}(x) :: x < 0$
Answer: $(\forall\ x)(x < 0)$ is `False`, but $(\exists x)(x < 0)$ is `True`.

(d) $\mathrm{P}(x) :: 2^x > 0$
Answer: $(\forall\ x)(2^x > 0)$ is `True` and $(\exists x)(2^x > 0)$ is `True`.

(e) Just a check that you are thinking: Can it be `True` that $(\forall\ x)(\mathrm{P}(x))$ is `True` and $(\exists\ x)(\neg\mathrm{P}(x))$ is `True`?
Answer: Well, no! If $\mathrm{P}(x)$ is `True` for all x's, then there is no x for which it is `False`.

(f) Just a check that you are thinking: Do any of your answers change is you replace the domain $\mathbb{R}$ with natural numbers, integers, or rational numbers: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.
Answer: Well, yes! The answers for part (a) remains the same, but (b) is always `False`, (c) is `False`, `False` for domain $\mathbb{N}$, but (c) is `False`, `True` for domain $\mathbb{Z}$ and $\mathbb{Q}$, and the answers for part (d) and (e) remain the same.

2. You know $(\forall\ x)(\forall\ y) \equiv (\forall\ y)(\forall\ x)$ and $(\exists\ x)(\exists\ y) \equiv (\exists\ y)(\exists\ x)$. But, in general $(\forall x)(\exists\ y) \not\equiv (\exists\ y)(\forall\ x)$

Put $(\forall\ x)(\forall\ y)$, $(\exists\ x)(\exists\ y)$, $(\forall x)(\exists\ y)$ and $(\exists\ y)(\forall\ x)$ in front of each of these preicates and decide if the quantified expressions are `True` or `False`. Assume the domain is the real numbers $\mathbb{R}$.

(a) $\mathrm{P}(x, y) :: x = y + 1$
Answer:

i. $(\forall\ x)(\forall\ y)(x = y + 1)$ is `False`.
ii. $(\exists\ x)(\exists\ y)(x = y + 1)$ is `True`.
iii. $(\forall\ x)(\exists\ y)(x = y + 1)$ is `True`. Given x let $y = x - 1$.
iv. $(\exists\ y)(\forall\ x)(x = y + 1)$ is `False`.

(b) $\mathrm{P}(x, y) :: x^2 - x - 1 = y$
Answer:

i. $(\forall\, x)(\forall\, y)(x^2 - x - 1 = y)$ is `False`.
ii. $(\exists\, x)(\exists\, y)(x^2 - x - 1 = y)$ is `True`.
iii. $(\forall\, x)(\exists\, y)(x^2 - x - 1 = y)$ is `True`.
iv. $(\exists\, y)(\forall\, x)(x^2 - x - 1 = y)$ is `False`.

(c) $\textsc{p}(x, y) :: x < y$
Answer:
i. $(\forall\, x)(\forall\, y)(x < y)$ is `False`.
ii. $(\exists\, x)(\exists\, y)(x < y)$ is `True`.
iii. $(\forall\, x)(\exists\, y)(x < y)$ is `True`.
iv. $(\exists\, y)(\forall\, x)(x < y)$ is `False`.

(d) $\textsc{p}(x, y) :: 2^x > y$
Answer:
i. $(\forall\, x)(\forall\, y)(2^x > y)$ is `False`.
ii. $(\exists\, x)(\exists\, y)(2^x > y)$ is `True`.
iii. $(\forall\, x)(\exists\, y)(2^x > y)$ is `True`.
iv. $(\exists\, y)(\forall\, x)(2^x > y)$ is `True`.

3. Know your logarithms. Compute:

(a) $\lg 256$
Answer: $\lg 256 = 8$ because $2^8 = 256$.

(b) $\lg 1/256$
Answer: $\lg 1/256 = -8$ because $2^{-8} = 1/256$.

(c) $\lg \sqrt[4]{32}$
Answer: $\lg \sqrt[4]{32} = 5/4$ because $\lg \sqrt[4]{32} = 0.25 \lg 32$, or $2^{5/4} = \sqrt[4]{2^5}$.

(d) $\lg (0.25\sqrt{2})$
Answer: $\lg (0.25\sqrt{2}) = -3/2$ because $\lg (0.25\sqrt{2}) = \lg 0.25 + \lg \sqrt{2} = -2 + \frac{1}{2}$.

(e) Write $\log_b x$ in terms of $\log_c x$.
Answer: The change of base rule for logarithms is

$$\log_b x = \frac{\log_c x}{\log_c b}$$

This is easy to derive: Let $y = \log_b x$ so that $b^y = x$. Compute the log base c of both sides and use the "log of a power is the power times the log" rule

$$\begin{aligned} b^y &= x \\ \log_c b^y &= \log_c x \\ y \log_c b &= \log_c x \end{aligned}$$

and solve for y.

4. Use Horner's rule to evaluate these polynomials at the given value of x.

(a) $p(x) = x^4 + x^3 + x^2 + x + 1$ at $x = 2$. (How is this related to binary numbers?)
Answer:

Horner's Rule @ $x = 2$				
1	1	1	1	1
	2	6	14	30
1	3	7	15	31

Therefore $p(2) = 31$. This number can be interpreted as the binary number $(11111)_2$.

(b) $p(x) = 5x^4 + 7x^2 + 8x + 1$ at $x = -1$.

Answer:

Horner's Rule @ $x = -1$				
5	0	7	8	1
	-5	5	-12	4
5	-5	12	-4	5

Therefore $p(-1) = 5$.

5. Use Horner's rule to convert the following numbers to their decimal equivalent.

(a) $(10101010)_2$

Answer:

Horner's Rule @ $x = 2$							
1	0	1	0	1	0	1	0
	2	4	10	20	42	84	170
1	2	5	10	21	42	85	170

Therefore, $(10101010)_2 = 170$

(b) $(1010.1010)_2$. Don't use *Horner's rule* again. Use what you just learned.

Answer: From the previous problem $(1010.1010)_2 = 170/16 = 10\frac{5}{8} = 10.625$

(c) $(BE)_{16}$

Answer:

Horner's Rule @ $x = 16$	
11	14
	176
11	190

Therefore, $(BE)_{16} = 190$

(d) $(B.E)_{16}$. Don't use *Horner's rule* again. Use what you just learned.

Answer: From the previous problem $(B.E)_{16} = 190/16 = 11\frac{7}{8} = 11.875$.

6. Use repeated remaindering to convert the following numbers to their binary equivalent.

(a) 237

Answer:

Repeated Remaindering mod 2

Quotients	237	118	59	29	14	7	3	1
Remainders	1	0	1	1	0	1	1	1

Therefore, $237 = (11101101)_2$.

(b) 2.37 (Expand to 5 bits after the binary point)

Answer: From the previous problem, you could divide $(11101101)_2$ by $100 = (1100100)_2$, but that is rather unnatural. So use two steps

i. Convert 2 to $(10)_2$.

ii. Convert 0.37 to binary by repeated mulitiplication by 2.

$$0.37 \times 2 = 0.74$$
$$0.74 \times 2 = 1.48$$
$$0.48 \times 2 = 0.96$$
$$0.96 \times 2 = 1.92$$
$$0.92 \times 2 = 1.84$$

Therefore, $2.37 = (10.01011\cdots)_2$. Eventually, the pattern will repeat. I wonder how far you need to go before the repetition starts.

Total Points: 0 2015-02-17 to 2015-02-20