Counting Cyclic Permutations

There is a relatively simple two-dimensional recurrence equation that counts cyclic permutations.

The recursion is known as Stirling's identity of the first kind.

It is similar to Pascal's identity for binomial coefficients, which counts subsets of a given cardinality.

Small Examples of Cyclic Permutations

By definition there is one permutation on the empty set.

It is the empty permutationBut let's look at non-empty sets.

Consider a 1 element set, $\{0\}$.

There is 1 permutation $\langle 0 \rangle = [0]$.

With a 2 element set, $\{0, 1\}$.

There is 1 permutation with 1-cycle $\langle 1, 0 \rangle = [1, 0]$.

And, 1 permutation with 2-cycles $\langle 0, 1 \rangle = [0][1]$.

Small Examples of Cyclic Permutations

It is convenient to collect the computations on the previous slide into a triangular array, much like Pascal's triangles for binomial coefficients.

n Cycle k

		k		
		0	1	2
	0	1		
n	1	0	1	
	2	0	1	1

- On no elements (row 0) there is 1 permutation.
- On 1 element (row 1) there is 1 permutation.
- On 2 elements (row 2) there is 1 permutation with 1-cycle and 1 permutation with 2-cycle

Recursion on Small Examples

Let's use the permutations on 2 elements to create permutations on 3 elements.

$$[0, 1] \longrightarrow [0, 1][2]$$

Permutation $[0, 1]$ creates a 2-cycle permutation on 3 elements.

$$\begin{tikzcd} {[0][1]} \arrow[r] \arrow[dr] & {[0][1, 2]} \\ & {[0, 2][1]} \end{tikzcd}$$

Permutation $[0][1]$ creates two 2-cycle permutation on 3 elements. These are the only 2-cycle permutations on 3 elements.

Permutations 3 Elements

Let's list the 6 permutations on the 3 element set $\{0, 1, 2\}$.

- 1-cycle permutations: $[0, 1, 2]$ and $[0, 2, 1]$
- 2-cycle permutations: $[0, 1][2]$, $[0, 2][1]$ and $[1, 2][0]$
- 3-cycle permutations: $[0][1][2]$

Let's add row 3 to Stirling's triangle of the first kind

	n Cycle k				Row
	0	1	2	3	Sum
0	1				1
1	0	1			1
2	0	1	1		2
3	0	2	3	1	6

Recursive Definition of Stirling Numbers of the First Kind

To write a recurrence equation for what you've learned from these small examples you need to name the numbers you've computed.

Let $\left[{n \atop k}\right]$ denote the number of k-cycle permutations of n elements. $\left[{n \atop k}\right]$ is pronounced "n cycle k."

You've computed

$$\begin{array}{llll}
\left[{0 \atop 0}\right] = 1 & & & \\
\left[{1 \atop 0}\right] = 0 & \left[{1 \atop 1}\right] = 1 & & \\
\left[{2 \atop 0}\right] = 0 & \left[{2 \atop 1}\right] = 1 & \left[{2 \atop 2}\right] = 1 & \\
\left[{3 \atop 0}\right] = 0 & \left[{3 \atop 1}\right] = 2 & \left[{3 \atop 2}\right] = 3 & \left[{3 \atop 3}\right] = 1
\end{array}$$

Recursive Definition of Stirling Numbers of the First Kind

Let's look at how you can use the two 1-cycle permutations on three elements to create 2-cycle permutations on four elements.

$$[0,\ 1,\ 2] \longrightarrow [0,\ 1,\ 2][3]$$

$$[0,\ 2,\ 1] \longrightarrow [0,\ 2,\ 1][3]$$

Each 1-cycle permutation creates a 2-cycle permutation on 4 elements.

Recursive Definition of Stirling Numbers of the First Kind

Look at how 2-cycle permutation on three elements create 2-cycle permutations on four elements.

$$
\begin{tikzcd}
{[0][1,\,2]} \arrow[rr] \arrow[rrd] \arrow[rrdd] & & {[0][1,\,2,\,3]} \\
& & {[0][1,\,3,\,2]} \\
& & {[0,\,3][1,\,2]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[1][0,\,2]} \arrow[rr] \arrow[rrd] \arrow[rrdd] & & {[1][0,\,2,\,3]} \\
& & {[1][0,\,3,\,2]} \\
& & {[1,\,3][0,\,2]}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[2][0,\,1]} \arrow[rr] \arrow[rrd] \arrow[rrdd] & & {[2][0,\,1,\,3]} \\
& & {[2][0,\,3,\,1]} \\
& & {[2,\,3][0,\,1]}
\end{tikzcd}
$$

Each 2-cycle permutation creates three 2-cycle permutation on 4 elements.

Recursive Definition of Stirling Numbers of the First Kind

Using the previous results you can conclude there are

$$3 \cdot 3 + 2 = 11$$

2-cycle permutations on 4 elements.

Each of the three 2-cycle permutations on 3 elements generated three 2-cycle permutations on 4 elements, and 2 were generated by 1-cycle permutations on 3 elements. In Stirling's notation

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Recursive Definition of Stirling Numbers of the First Kind

The general case for recursion for Stirling numbers of the first kind is

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

Using this recurrence, you can compute values of Stirling numbers of the first kind. Recall, you already know

	n Cycle k						Row
	0	1	2	3	4	5	Sum
0	1						1
1	0	1					1
2	0	1	1				2
3	0	2	3	1			6
4	0	6	11	6	1		24
5	0	24	50	35	10	1	120

Explicit Formula for Stirling Numbers of the First Kind

There is an explicit formula for Stirling numbers of the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

The sum of all values in row n of Stirling's triangle of the first kind count the number of permutations on n elements. Of course, these sums are equal to $n!$.

Problems on Counting Cyclic Permutations

Show your understanding of this topic by completing the problems found at Counting Cyclic Permutations