**Counting Functions by Subsets**

It is useful to be able to count the number of functions from a finite set to another finite set.

One way to count functions is by recognizing that a function can be represented as a set of ordered pairs with a special property

$$(\forall x \in \mathbb{X})(\exists! y \in \mathbb{Y})((x, y) \in f)$$

Let $f : \mathbb{X} \to \mathbb{Y}$ be a function. Let $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$.

Then $f$ is a subset of the Cartesian product $\mathbb{X} \times \mathbb{Y}$. And the cardinality of $f$ is $n$.

For each of the $n$ pairs $(x, \sqcup) \in f$ there are $m$ possible ways to fill in the $y$ value. Therefore, there are

$$m^n \quad \text{functions } f : \mathbb{X} \to \mathbb{Y}$$

**Counting Functions by Subsets**

As a "counting functions" example, let

$$\mathbb{X} = \{0, 1\} \quad \text{and} \quad \mathbb{Y} = \{a, b, c\}$$

There are $3^2 = 9$ functions from $\mathbb{X}$ to $\mathbb{Y}$.

1. $\{(0, a), (1, a)\}$
2. $\{(0, a), (1, b)\}$
3. $\{(0, a), (1, c)\}$
4. $\{(0, b), (1, a)\}$
5. $\{(0, b), (1, b)\}$
6. $\{(0, b), (1, c)\}$
7. $\{(0, c), (1, a)\}$
8. $\{(0, c), (1, b)\}$
9. $\{(0, c), (1, c)\}$

**Counting Functions by Adjacency Matrix**

Another way to count functions is by recognizing that a function can be represented by an adjacency matrices. Let $f : \mathbb{X} \to \mathbb{Y}$ be a function, and let $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$. Then $f$ is an $n \times m$ adjacency matrix.

| | | $m$ columns | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | $\cdots$ | $m-2$ | $m-1$ |
| | 0 | | | | | | |
| $n$ | 1 | | | | | | |
| rows | $\vdots$ | | | | | | |
| | $n-1$ | | | | | | |

Each row will have a 1 in one and only one column. There are $m$ column choices for each of the $n$ rows.

Therefore, there are $m^n$ functions from $\mathbb{X}$ to $\mathbb{Y}$.

**Counting Functions by Adjacency Matrix**

As a "counting functions" example, let

$$\mathbb{X} = \{0, 1\} \quad \text{and} \quad \mathbb{Y} = \{a, b, c\}$$

There are $3^2 = 9$ functions from $\mathbb{X}$ to $\mathbb{Y}$.

1. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}$

2. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}$

3. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}$

4. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}$

5. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}$

6. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$

7. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}$

8. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}$

9. $\begin{array}{c|ccc} & a & b & c \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}$

**Counting Functions by Bipartite Graph**

Another way to count functions is by recognizing that a function can be represented by a bipartite graphs.

Let $f : \mathbb{X} \to \mathbb{Y}$ be a function. Let $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$.

Then, $f$ is a bipartite graph from $\mathbb{X}$ to $\mathbb{Y}$.

A bipartite graph is a collection of directed edges from $\mathbb{X}$ to $\mathbb{Y}$.

To be a function, the graph has one and only one edge leaving each element in $\mathbb{X}$.

That edge can be directed at any of the $m$ elements in $\mathbb{Y}$.

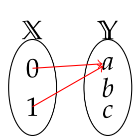
Therefore, there are $m^n$ functions from $\mathbb{X}$ to $\mathbb{Y}$.

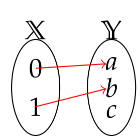
**Counting Functions by Adjacency Matrix**

As a "counting functions" example, let

$$\mathbb{X} = \{0, 1\} \quad \text{and} \quad \mathbb{Y} = \{a, b, c\}$$

There are $3^2 = 9$ functions from $\mathbb{X}$ to $\mathbb{Y}$.

1.



2.
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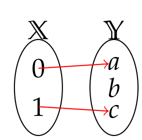

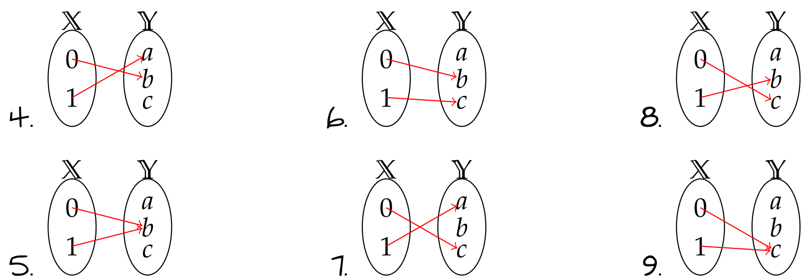


**Counting Functions**

**Theorem 1** (Counting Functions)**.** *Let the cardinality of $\mathbb{X}$ be $n$ ($|\mathbb{X}| = n$) and let the cardinality of $\mathbb{Y}$ be $n$ ($|\mathbb{Y}| = m$).*

*Then there are*

$$m^n \quad \textit{functions from } \mathbb{X} \textit{ to } \mathbb{Y}$$