One-to-One Functions

Definition 1 (One-to-One Function)**.** A function $f : \mathbb{X} \to \mathbb{Y}$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, or equivalently, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Each y in $\mathbb{Y}$, is the image of at most one x in $\mathbb{X}$.

$$(\forall x_1, x_2 \in \mathbb{Y})((f(x_1) = f(x_2)) \to (x_1 = x_2))$$

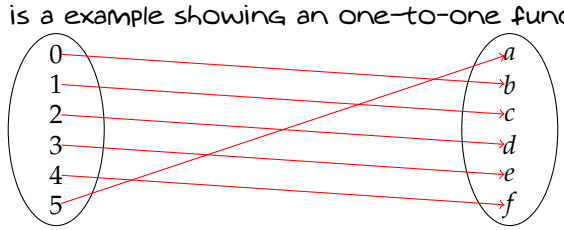
or

$$(\forall x_1, x_2 \in \mathbb{Y})((x_1 \neq x_2) \to (f(x_1) \neq f(x_2)))$$

One-to-One Functions

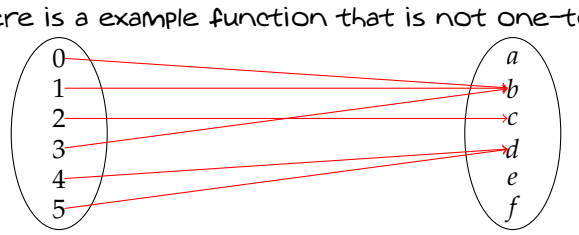
Recall, that a function can be represented by a graph, a good way to visualize the one-to-one idea.

Here is a example showing an one-to-one function.



One-to-One Functions

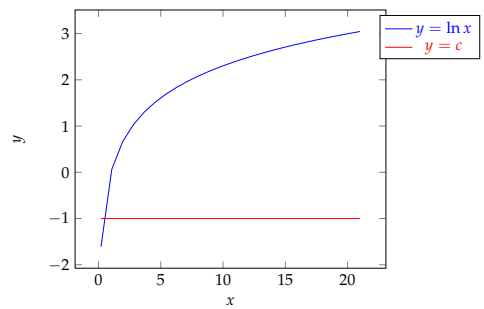
And, here is a example function that is not one-to-one.



b is the images of three values in $\mathbb{X}$ and d is the images of both 4 and 5.

The Horizontal Line Test for One-to-One-ness

When a function $y = f(x)$ is graphed using Cartesian coordinates in the traditional analytic geometry fashion, the function is one-to-one when every horizontal line crosses its graph at most once.



When some horizontal line crosses the graph of $y = f(x)$, more than once the f is not one-to-one.

Example One-to-One Functions

The following are one-to-one functions.

- All polynomials of degree 1 are one-to=one. For instance, $f(x) = 3x + 2$ is one-to-one.

  To see this, if $3x_1 + 2 = 3x_2 + 2$, then $x_1 = x_2$.

- The common (Base 10) logarithm $f(x) = \log x$ is one-to-one. To see this, if $\log x_1 = \log x_2$, then $\log x_1 - \log x_2 = 0$ and so $\log(x_1/x_2) = 0$ which implies $x_1/x_2 = 1$.

Example Functions That Are Not One-to-One

The following are not one-to-one functions.

- $f(x) = 3x^2 + 2$ is not one-to-one.

  To see this, notice $f(-1) = 5$ and $f(1) = 5$.

- $f(n) = n \bmod 3$ is not one-to-one. $1 \bmod 3 = 4 \bmod 3 = 7 \bmod 3 = \cdots$.

One-to-One Depends on Cardinality of Domain and Co-domain

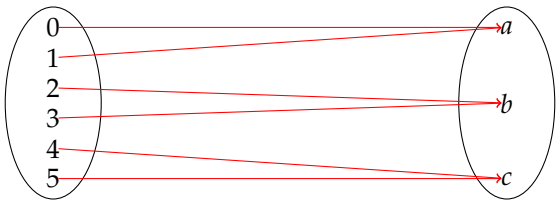
Let $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$ be the cardinalities of $\mathbb{X}$ and $\mathbb{Y}$.

Recall, the <u>cardinality</u> is the number of elements in the set.

If $\mathbb{Y}$ has fewer elements than $\mathbb{X}$, that is, if $|\mathbb{Y}| < |\mathbb{X}|$, then it is <u>not possible</u> to define an one-to-one (total) function from $\mathbb{X}$ to $\mathbb{Y}$.

No One-to-One Functions from Large to Smaller Sets

You cannot have an one-to-one function from a large set to a smaller set.



On the other hand, a small domain does not guarantee a one-to-one function.