Polynomials

Much of algebra focuses on the subject of polynomials.

This is because:

- Polynomials model many simple systems. In particular, linear and quadratic systems.
- Polynomials are easy to evaluate.
- Many complex systems can be well approximated by polynomials.

Basic Ideas about Polynomials

Constant polynomials have the form $p(x) = c$. For instance, $p(x) = 7$ is a constant polynomial.

Linear polynomials have the form $p(x) = mx + b$. For instance, $p(x) = 3x + 2$ is a linear polynomial.

Quadratic polynomials have the form $p(x) = ax^2 + bx + c$. For instance, $p(x) = x^2 - x - 1$ is a quadratic polynomial.

Basic Ideas about Polynomials

The degree of a polynomial is the value of its largest exponent.

For instance, The degree of $p(x) = 7$ is 0, the degree of $p(x) = 3x + 2$ is 1, and the degree of $p(x) = x^2 - x - 1$ is 2.

A root, r, of a polynomial $p(x)$ is a value of x such that $p(r) = 0$.

For instance, $p(x) = 7$ has no roots, the root of $p(x) = 3x + 2$ is $r = -2/3$, and the roots of $p(x) = x^2 - x - 1$ are

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad \text{and} \quad \overline{\varphi} = \frac{1-\sqrt{5}}{2} \approx -0.618$$

The Power Basis

The powers of x:

$$1, \quad x, \quad x^2, \quad x^3, \quad x^4, \quad \ldots$$

form a basis for polynomials.

That is, every polynomial $p(x)$ can be written as a sum of powers of x.

$$p(x) = \sum_{k=0}^{n-1} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

The Falling Factorial Power Basis

Falling factorial powers

$$x^{\underline{0}} = 1, x^{\underline{1}} = x, x^{\underline{2}} = x(x-1), x^{\underline{3}} = x(x-1)(x-2)$$

$$x^{\underline{4}} = x(x-1)(x-2)(x-3),\ x^{\underline{5}} = x(x-1)(x-2)(x-3)(x-4), \ldots$$

form an alternative basis that better models some systems.

There are many bases for the polynomials, but this is beyond the scope of this class.

The Binomial Theorem

One of the classic results from algebra is the binomial theorem.

Theorem 1 (Binomial Theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

In more detail

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + y^n$$

The Binomial Theorem

Instances of the binomial theorem include:

$$(x+y)^0 = 1$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Notice coefficient pattern 1, 1 1, 1 2 1, 1 3 3 1 come from Pascal's triangle.

Geometric Polynomials

The polynomials

$$1, \quad 1+x, \quad 1+x+x^2, \quad 1+x+x^2+x^3, \quad \ldots$$

are <u>geometric sums</u>.

Theorem 2 (Geometric Sum)**.** *Given $x \neq 1$, the geometric sum*

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

For instance,

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

$$1 + 10 + 100 + 1000 + \cdots + 10^{n-1} = \frac{10^n - 1}{9}$$