Predicate Logic

Predicate logic extends Boolean logic in three ways:

1. There is a domain of discourse that provides values that can substituted into statements. A generic symbol for this domain is $\mathbb{U}$, a universal set.

2. The truth value of a predicate statement depends on the value of one or more variables from the domain.

3. Predicate statements are quantified.

Predicate Notation

Let $\mathsf{p}(x)$ denote a predicate of one variable x.

And let $\mathsf{q}(x, y)$ and $\mathsf{r}(x, y)$ denote predicates on two variables x and y.

Let $\doteq$ denote how a predicate is defined.

For instance

$$\mathsf{p}(x) \doteq x \text{ is a cat.} \quad x \in \mathbb{U} = \text{set of animals.}$$

$$\mathsf{q}(x, y) \doteq x \text{ is a } y$$

$$\mathsf{r}(x, y) \doteq x \text{ can fool } y$$

Example Predicate Statements

The following statements are predicates on the given domain.

- $\mathsf{p}(x) \doteq (x = \texttt{False})$ on the domain of bits.

- $\mathsf{q}(x, y) \doteq (x - y = 0)$ on the domain of real numbers.

- $\mathsf{r}(a, n, q, r) \doteq (a = nq + r)$ on the domain of integers.

The truth of these equations depends on the values of the variables.

Example Predicate Statements

Consider assigning values to the variables in the predicates on the previous slide.

- Let $\mathsf{p}(x) \doteq (x = \texttt{False})$.

$$\mathsf{p}(\texttt{False}) = \texttt{True} \quad \text{and} \quad \mathsf{p}(\texttt{True}) = \texttt{False}.$$

- Let $\mathsf{Q}(x, y) \doteq (x - y = 0)$.

$$\mathsf{Q}(x, x) = \texttt{True} \quad \text{and} \quad \text{when } x \neq y \quad \mathsf{Q}(x, y) = \texttt{False}$$

- Let $\mathsf{r}(a, n, q, r) \doteq (a = nq + r)$.

$$\mathsf{r}(34, 7, 4, 6) = \texttt{True} \quad \text{and} \quad \mathsf{r}(34, 7, 4, 5) = \texttt{False}.$$

Quantifiers

A predicate statement $\mathsf{p}(x)$ can be

- True for all values of x.
- False for all values of x.
- True for some values of x.
- False for some values of x.

For all and for some are quantifiers.

For All: The Universal Quantifier

Consider the predicate statement $\mathsf{p}(x) \doteq (x^2 \geq 0)$ on the set of real numbers.

$\mathsf{p}(x)$ is True for all values of x.

For every real number x, x squared is greater than or equal to 0.

Using mathematical symbols you would write

$$(\forall x \in \mathbb{R})(x^2 \geq 0)$$

The upside-down A is read "for all."

For All: The Universal Quantifier

Consider the predicate statement $p(x) \doteq (x^2 < 0)$ on the set of real numbers.

$p(x)$ is `False` <u>for all</u> values of x.

<u>For all</u> values of x, not $p(x)$ is `True`.

$$(\forall x \in \mathbb{R})(x^2 \not< 0) \equiv (\forall x \in \mathbb{R})(x^2 \geq 0)$$

Another way to say this is:

There is no real number x such that $x^2 < 0$.

There Exists: The Existential Quantifier

Consider the predicate statement $p(x) \doteq x^2 - x - 1 = 0$ on the set of real numbers.

$p(x)$ is `True` <u>for some</u> values of x.

You can compute the roots of the equation by using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{5}}{2}$$

There are two real numbers that satisfy $x^2 - x - 1 = 0$.

Using mathematical symbols you would write

$$(\exists x \in \mathbb{R})(x^2 - x - 1 = 0)$$

The backwards $\exists$ is read "there exists."

There Exists: The Existential Quantifier

Consider the predicate statement $p(x) \doteq x^2 - x - 1 = 0$ on the set of real numbers.

$p(x)$ is `False` <u>for some</u> values of x.

You can demonstrate this by <u>counterexample</u>.

For instance, $x = 1$ does not satisfy $x^2 - x - 1 = 0$.

There are real numbers x such that $x^2 - x - 1 \neq 0$.

Using mathematical symbols you would write

$$(\exists x \in \mathbb{R})(x^2 - x - 1 \neq 0)$$

Another way to say this is:

It is not the case that $x^2 - x - 1 = 0$ for all values of x.

Existence and Uniqueness

Uniqueness is an important property.

In many situations we would like to know two things.

1. A thing exists that satisfies a given predicate, the problem has a "solution."

2. There is only one thing that satisfies the property, the "solution" is unique.

Because uniqueness is important, syntactic sugar can be applied to the existential quantifier to signify uniqueness.

$$(\exists!\ x \in \mathbb{R})(x^2 - 4x + 4 = 0)$$

The exclamation ! indicates there is one and only one x, $x = 2$, that satisfies the equation.