De Morgan Like Predicate Laws

Just as there are laws for Boolean logic and sets, there are laws for th use predicate statements.

1. De Moran-like laws for distributing NOT.
2. Commutative-like laws for re-ordering of quantifiers.

De Morgan Like Predicate Laws

There are relationships among negation $\neg$, and quantifiers for all $\forall$ and there exists $\exists$, that are similar to De Morgan's Laws.

If $\mathtt{P}(x)$ is not `True` for all values of x, then there exist an x such that $\mathtt{P}(x)$ is `False`, written

$$\neg(\forall x)(\mathtt{P}(x)) \equiv (\exists x)(\neg \mathtt{P}(x))$$

Similarly, If there exists does not exist an x such that $\mathtt{P}(x)$ is `True` then for all values of x, $\mathtt{P}(x)$ is `False`.

$$\neg(\exists x)(\mathtt{P}(x)) \equiv (\forall x)(\neg \mathtt{P}(x))$$

De Morgan Like Predicate Laws

Let's walk through a mathematical example. Consider:

$$(\forall x \in \mathbb{R})(x^2 \geq 0)$$

This says, x^2 is greater than or equal to 0 for every real number x. A `True` statement.

The negation of $x^2 \geq 0$ is $x^2 < 0$.

The statement: There is a real number x such $x^2 < 0$ is `False`.

$$(\exists x \in \mathbb{R})(x^2 < 0) = \mathtt{False}$$

Therefore,

$$\neg(\exists x \in \mathbb{R})(x^2 < 0) = \mathtt{True}$$

Commutative Laws for Quantifiers

Consider a two variable predicate $\mathsf{P}(x, y)$, say

$$\mathsf{P}(x, y) \doteq x \text{ can fool } y$$

The statement

$$(\forall x)(\forall y)(\mathsf{P}(x, y))$$

says every person x can fool every person y, in particular x can fool himself or herself.

We can commute the two quantifiers.

$$(\forall y)(\forall x)(\mathsf{P}(x, y))$$

Which says every person y can be fooled by every person x

The two statement are equivalent.

$$(\forall x)(\forall y)(\mathsf{P}(x, y)) \equiv (\forall y)(\forall x)(\mathsf{P}(x, y))$$

For All and For All Commute

Consider

$$(\forall x)(\forall y)(x \text{ can fool } y)$$

says every person x, Sally for example, can fool every person y, for example Ed.

$$(\forall y)(\forall x)(x \text{ can fool } y)$$

says every person y, Ed for example, can be fooled by every person x, in particular by Sally.

$$(\forall x)(\forall y)(\mathsf{P}(x, y)) \equiv (\forall y)(\forall x)(\mathsf{P}(x, y))$$

Commutative Laws for Quantifiers

Again consider $\mathsf{P}(x, y) \doteq x$ can fool y.

The statement

$$(\exists x)(\exists y)(\mathsf{P}(x, y))$$

says someone x can fool someone y.

Commute the two quantifiers.

$$(\exists y)(\exists x)(\mathsf{P}(x, y))$$

This says someone y can be fooled by someone x.

The two statement are equivalent.

$$(\exists x)(\exists y)(\mathsf{P}(x, y)) \equiv (\exists y)(\exists x)(\mathsf{P}(x, y))$$

There Exists and There Exists Commute

Consider

$$(\exists x)(\exists y)(x \text{ can fool } y)$$

says there is a person x, call her Sally, who can fool some person y, call him Ed.

$$(\exists y)(\exists x)(x \text{ can fool } y)$$

says there is a person y, say Ed, who can be fooled by some person x, perhaps by Sally.

$$(\exists x)(\exists y)(\mathsf{P}(x, y)) \equiv (\exists y)(\exists x)(\mathsf{P}(x, y))$$

Commutative Laws for Quantifiers

Again consider $\mathsf{P}(x, y) \doteq x$ can fool y.

The statement

$$(\forall x)(\exists y)(\mathsf{P}(x, y))$$

says everyone can fool someone, not necessarily the same someone.

Commute the two quantifiers.

$$(\exists y)(\forall x)(\mathsf{P}(x, y))$$

This says there is someone who can be fooled by everyone.

The two statement are not equivalent.

$$(\forall x)(\exists y)(\mathsf{P}(x, y)) \not\equiv (\exists y)(\forall x)(\mathsf{P}(x, y))$$

There Exists and For All Do Not Commute

Consider

$$(\forall x)(\exists y)(x \text{ can fool } y)$$

says everyone can fool someone. Maybe, Sally can fool Ed, and Ed can fool John.

$$(\exists y)(\forall x)(x \text{ can fool } y)$$

says someone can be fooled by everyone. Perhaps it is Sally who can be fooled by everyone.

$$(\forall x)(\exists y)(\mathsf{P}(x, y)) \not\equiv (\exists y)(\forall x)(\mathsf{P}(x, y))$$

There Exists and For All Do Not Commute

Let's walk through a mathematical example. Consider:

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x < y)$$

This says for every real number x, there is a real number y such that $x < y$. Do you agree this is clearly `True`?

Now consider:

$$(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(x < y)$$

This says there is a real number y, that is larger than every real number x. Do you agree this is clearly <u>not</u> `True`?