Partitions of a Set

Pretend you are given the set $\mathbb{U} = \{a, b, c, d\}$.

You can partition $\mathbb{U}$ into two subsets in multiple ways.

1. $\mathbb{U} = \{a, b, c, d\} = \{a\} \cup \{b, c, d\}$.
2. $\mathbb{U} = \{a, b, c, d\} = \{b\} \cup \{a, c, d\}$.
3. $\mathbb{U} = \{a, b, c, d\} = \{c\} \cup \{a, b, d\}$.
4. $\mathbb{U} = \{a, b, c, d\} = \{d\} \cup \{a, b, c\}$.
5. $\mathbb{U} = \{a, b, c, d\} = \{a, b\} \cup \{c, d\}$.
6. $\mathbb{U} = \{a, b, c, d\} = \{a, c\} \cup \{b, d\}$.
7. $\mathbb{U} = \{a, b, c, d\} = \{a, d\} \cup \{b, c\}$.

There are 7 two subset partitions of a 4 element set.

Partitions of a Set

Definition 1 (Partition of a Set)**.** Let $\mathbb{U}$ be a set. A partition of $\mathbb{U}$ is a set of subsets

$$\{\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2, \ldots \mathbb{X}_{n-1}\}$$

such that

- Each subset $\mathbb{X}_k$ is non-empty, $(\forall k \in \mathbb{Z}_n)(\mathbb{X}_k \neq \emptyset)$
- The subsets are mutually exclusive,

$$(\forall k, j \in \mathbb{Z}_n)((k \neq j) \rightarrow (\mathbb{X}_k \cap \mathbb{X}_j = \emptyset))$$

- The subsets cover $\mathbb{U}$, $(\mathbb{U} = \mathbb{X}_0 \cup \mathbb{X}_1 \cup \mathbb{X}_2 \cup \cdots \cup \mathbb{X}_{n-1})$

The subsets $\mathbb{X}_k$ are called equivalence classes.

Why Study Partitions?

Collecting and classifying objects based on a common shared property is a useful task.

When such a collection is a partition the classes are independent or orthogonal, they do not share any elements.

Therefore, operations on the classes are independent.

Decreasing dependency is an important software design concept.

Coupling is a software metric that measures the dependency of programming code.

Describing how the coupling function is defined is beyond the scope of this course.

Counting Partitions

You've learned how to count subsets of a set.

Let $\mathbb{U}$ be an n-element set.

- There are 2^n subsets of $\mathbb{U}$.
- There are
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
subsets of $\mathbb{U}$ with k elements.

You can count partitions by using ideas similar to those that led to Pascal's triangle and binomial coefficients.

Counting Partitions

Let $\mathbb{U}$ be an n-element set.

In how many ways can $\mathbb{U}$ be partitioned into k subsets?

Several slides back we saw there are 7 two subset partitions of a 4 element set.

To denote this fact, write

$$\left\{ {4 \atop 2} \right\} = 7$$

and say "4 subset 2 equals 7."

The Stirling number of the second kind

$$\left\{ {n \atop k} \right\} \qquad (n \text{ subset } k)$$

counts the number of partitions of an n-element set into k equivalence classes.

Stirling's Numbers of the Second Kind

Later in the course we will study Stirling's numbers of the second kind in more detail.

Here we only mention that partitions can be counted using two dimensional recurrence equation similar to Pascal's identity.

Stirling's identity of the second kind is

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \tag{1}$$

Stirling's Numbers of the Second Kind

The triangle that counts partitions is displayed below.

Stirling Numbers of the Second Kind $\left\{ {n \atop k} \right\}$

		Subset k								
		0	1	2	3	4	5	6	7	8
	0	1	0	0	0	0	0	0	0	0
	1	0	1	0	0	0	0	0	0	0
	2	0	1	1	0	0	0	0	0	0
	3	0	1	3	1	0	0	0	0	0
n	4	0	1	7	6	1	0	0	0	0
	5	0	1	15	25	10	1	0	0	0
	6	0	1	31	90	65	15	1	0	0
	7	0	1	63	301	350	140	21	1	0
	8	0	1	127	966	1701	1050	266	28	1