CSE 1400 Applied Discrete Mathematics
Logic & Sets

Department of Computer Sciences

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Problems on Boolean Logic

1. (Student will be able to identify the symbols used in Boolean logic.) Match the term with its symbol.

   (a) (a) AND
   (b) (c) Equivalent
   (c) (b) Exclusive or
   (d) (h) `True`
   (e) (e) `False`
   (f) (g) Implies (if ... then ...)
   (g) (d) NOT
   (h) (f) OR
   (i) (g) Conditional (if ... then ...)

   (a) $\wedge$
   (b) $\oplus$
   (c) $\equiv$
   (d) $\neg$
   (e) 0
   (f) $\vee$
   (g) $\rightarrow$
   (h) 1

2. (Student will be able to compute fundamental operators on Boolean variables) Fill in the following truth tables.

   (a) Not (Negation)

Input	Output
P	$\neg$P
0	1
1	0

   (b) And (Conjunction)

Input		Output
P	Q	P $\wedge$ Q
0	0	0
0	1	0
1	0	0
1	1	1

   (c) Or (Disjunction)

Input		Output
P	Q	P $\vee$ Q
0	0	0
0	1	1
1	0	1
1	1	1

   (d) Conditional (Implies, If ..., then ..., else `True`)

Input		Output
P	Q	$P \rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

(e) Not Conditional (Does Not Imply, $P \wedge \neg Q$)

Input		Output
P	Q	$P \not\rightarrow Q$
0	0	0
0	1	0
1	0	1
1	1	0

(f) Equivalent (Bi-Conditional)

Input		Output
P	Q	$P \equiv Q$
0	0	1
0	1	0
1	0	0
1	1	1

(g) XOR (Exclusive Or)

Input		Output
P	Q	$P \oplus Q$
0	0	0
0	1	1
1	0	1
1	1	0

(h) NAND (Not And)

Input		Output
P	Q	$\neg(P \wedge Q)$
0	0	1
0	1	1
1	0	1
1	1	0

(i) Nor (Not Or)

Input		Output
P	Q	$\neg(P \vee Q)$
0	0	1
0	1	0
1	0	0
1	1	0

3. Answer `True` or `False`.

(a) The Not operator can be written as

$$\neg P = \begin{cases} 1 & \text{if } P = 0 \\ 0 & \text{if } P = 1 \end{cases}$$

Answer: This is `True`.

(b) The And operator can be written as

$$P \wedge Q = \begin{cases} 1 & \text{if } (P = 1 \text{ and } Q = 1) \\ 0 & \text{otherwise} \end{cases}$$

Answer: This is `True`.

(c) The Or operator can be written as

$$P \vee Q = \begin{cases} 0 & \text{if } (P = 0 \text{ and } Q = 0) \\ 1 & \text{otherwise} \end{cases}$$

Answer: This is `True`.

(d) The Conditional operator can be written as

$$P \to Q = \begin{cases} 0 & \text{if } (P = 1 \text{ and } Q = 0) \\ 1 & \text{otherwise} \end{cases}$$

Answer: This is `True`.

4. It is useful and interesting to think of bits as numbers and to define arithmetic operations on them.

(a) The summation of two bits A and B produces a sum bit S and a carry bit C. Fill in the table that describes 1-bit summation.

Input		Output	
A	B	C	S
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

(b) What Boolean function $s(A, B)$ computes the sum bit S?
Answer: The Exclusive-Or operation computes S. Notice that Exclusive-Or can be written using And and Or

$$A \oplus B \equiv (\neg A \wedge B) \vee (A \wedge \neg B)$$

(c) What Boolean function $c(A, B)$ computes the carry bit C?
Answer: The And operation computes C.

(d) Extending addition to strings of bits requires defining addition of three bits at once: A, B, and the "carry-in" bit C_{in}, which is the "carry-out" bit C_{out} from the previous addition.

$$A + B + C_{in} = (S, C_{out})$$

Fill in the table that described this full-adder.

Input			Output	
C_{in}	A	B	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

(e) What Boolean function $s(A, B, C_{in})$ computes the sum bit S?
Answer: The sum bit is computed by the expression

$$S = A \oplus B \oplus C_{in}$$

Notice that this expression can be written as

$$s = (\neg A \wedge \neg B \wedge C_{in}) \vee (\neg A \wedge B \wedge \neg C_{in}) \vee (A \wedge \neg B \wedge \neg C_{in}) \vee (A \wedge B \wedge C_{in})$$

(f) What Boolean function $c(A, B, C_{in})$ computes the carry-out bit C_{out}?
Answer: The carry-out bit C_{out} can be computed by the expression

$$C_{out} = (A \wedge B) \vee (C_{in} \wedge (A \oplus B))$$

Notice that this expression can be written as

$$C_{out} = (\neg A \wedge B \wedge C_{in}) \vee (A \wedge \neg B \wedge C_{in}) \vee (A \wedge B \wedge \neg C_{in}) \vee (A \wedge B \wedge C_{in})$$

5. Which of the following are `True` and which are `False`?

(a) (P and Q) is equivalent to (Q and P), that is $(P \wedge Q) \equiv (Q \wedge P)$.
Answer: This is `True`. It is the commutative rule for the And operation. There is a similar rule for Or.

(b) (P and (Q and R)) is equivalent to ((P and Q) and R), that is $(P \wedge (Q \wedge R)) \equiv ((P \wedge Q) \wedge R)$.
Answer: This is `True`. It is the associative rule for the And operation. There is a similar rule for the Or operation.

(c) (P and (Q or R)) is equivalent to ((P and Q) or (P and R), that is $(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$.
Answer: This is `True`. It is the distributive rule for And over Or operation. The similar rule is

$$(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$$

for distributing Or over And is also `True`.

(d) Not (P Or Q) is equivalent to (Not P) and (Not Q), that is $(\neg(P \vee Q)) \equiv (\neg P \wedge \neg Q)$.
Answer: This is `True`. It is one of De Morgan's rules for distributing Not over Or operation. The similar rule is

$$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$$

for distributing Not over And is also `True`.

(e) Not (P And Q) is equivalent to (Not P) Or (Not Q), that is $(\neg(P \wedge Q)) \equiv (\neg P \vee \neg Q)$.
Answer: This is `True`. It is one of De Morgan's rules for distributing Not over Or operation. The similar rule is

$$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$$

for distributing Not over And is also `True`.

(f) In problem 4e we wrote

$$A \oplus B \oplus C_{in}$$

without parenthesis. Is it `True` that Exclusive-Or is associative, that is,

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

so that parenthesis are not needed?

Answer: It is `True`. Examine the truth table

Input			Output				
A	B	C	$A \oplus$	$(B \oplus C)$	$\equiv$	$(A \oplus B)$	$\oplus C$
0	0	0	0	0	1	0	0
0	0	1	1	1	1	0	1
0	1	0	1	1	1	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	1	1
1	0	1	0	1	1	1	0
1	1	0	0	1	1	0	0
1	1	1	0	1	1	1	0

6. Given that $P = \texttt{True} = 1$, $Q = \texttt{False} = 0$ and $R = \texttt{False} = 0$, compute the value of the following propositions.

   (a) $\neg P \vee R$
   Answer: $\neg P \vee R = \neg\texttt{True} \vee \texttt{False} = \texttt{False}$

   (b) $\neg P \rightarrow R$
   Answer: $\neg P \rightarrow R = \neg\texttt{True} \rightarrow \texttt{False} = \texttt{True}$

   (c) $\neg(P \wedge Q) \vee \neg R$
   Answer: $\neg(P \wedge Q) \vee \neg R = \neg(\texttt{True} \wedge \texttt{False}) \vee \neg\texttt{False} = \texttt{True}$

   (d) $(P \rightarrow R) \rightarrow Q$
   Answer: $(P \rightarrow R) \rightarrow Q = (\texttt{True} \rightarrow \texttt{False}) \rightarrow \texttt{False} = \texttt{True}$.

   (e) $P \wedge \neg R$
   Answer: $P \wedge \neg R = \texttt{True} \wedge \neg\texttt{True} = \texttt{True}$

   (f) $(P \wedge Q) \rightarrow R$
   Answer: $(P \wedge Q) \rightarrow R(\texttt{True} \wedge \texttt{False}) \rightarrow \texttt{False} = \texttt{True}$

   (g) $\neg(P \vee \neg Q) \wedge \neg R$
   Answer: $\neg(P \vee \neg Q) \wedge \neg R\neg(\texttt{True} \vee \neg\texttt{False}) \wedge \neg\texttt{False} = \texttt{True}$

   (h) $P \rightarrow (R \rightarrow Q)$
   Answer: $P \rightarrow (R \rightarrow Q)\texttt{True} \rightarrow (\texttt{False} \rightarrow \texttt{False}) = \texttt{True}$

7. Express the conditional operator

$$P \rightarrow Q = \begin{cases} 1 & \text{if } P = 0 \text{ or } (P = 1 \text{ and } Q = 1) \\ 0 & \text{if } P = 1 \text{ and } Q = 0 \end{cases}$$

   using the not $\neg$ and or $\vee$ operations.
   Answer: $(P \rightarrow Q) \equiv (\neg P \vee Q)$.

8. Write the expression

```
if p then q else r
```

as a Boolean expression using variables P, Q and R and operators $\neg$, $\wedge$ and $\rightarrow$.

Answer: The code statement is equivalent to the statement

$$(P \rightarrow Q) \wedge (\neg P \rightarrow R)$$

This interpretation can be represented by the truth table below which shows the output is R when P is `False` and Q when P is `True`.

Input			Output		
P	Q	R	$((P \rightarrow Q)$	$\wedge$	$(\neg P \rightarrow R))$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	1	1	1

9. When P is `False`, describe the output value of the conditional from problem 8.

   Answer: When $P = 0$, the output is identical to R.

10. When P is `True`, describe the output value of the conditional from problem 8.

    Answer: When $P = 1$, the output is identical to Q.

11. Show that the conditional $P \rightarrow Q$ is equivalent to $\neg P \vee Q$.

    Answer:

Input		Output				
P	Q	$P \rightarrow Q$	$\equiv$	$\neg P$	$\vee$	Q
0	0	1	1	1	1	0
0	1	1	1	1	1	1
1	0	0	1	0	0	0
1	1	1	1	0	1	1

12. Show that the conditional $P \rightarrow Q$ is equivalent to its contra-positive $\neg Q \rightarrow \neg P$.

    Answer:

Input		Output			
P	Q	$P \rightarrow Q$	$\equiv$	$\neg Q \rightarrow \neg P$	
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	1	1	0
1	1	1	1	1	1

13. Construct truth tables to prove De Morgan's laws

(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Answer: The values in the blue columns are compared, and their equivalence is marked in crimson.

Input		Output					
p	q	$\neg$	$(p \wedge q)$	$\equiv$	$(\neg p$	$\vee$	$\neg q)$
0	0	1	0	1	1	1	1
0	1	1	0	1	1	1	0
1	0	1	0	1	0	1	1
1	1	0	1	1	0	0	0

(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Answer: The values in the blue columns are compared, and their equivalence is marked in crimson.

Input		Output					
p	q	$\neg$	$(p \vee q)$	$\equiv$	$(\neg p$	$\wedge$	$\neg q)$
0	0	1	0	1	1	1	1
0	1	0	1	1	1	0	0
1	0	0	1	1	0	0	1
1	1	0	1	1	0	0	0

14. Show how to distribute NOT into a conditional.

$$\neg(p \to q) = (\_\_\_\_\_)$$

Answer: By the equivalence $(p \to q) \equiv (\neg p \vee q)$ and by De Morgan's law

$$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv (p \wedge \neg q)$$

15. Show how to factoring NOT out of a conditional.

$$\neg p \to \neg q \equiv \neg(\_\_\_\_\_)$$

Answer: By the equivalence $(\neg p \to \neg q) \equiv (p \vee \neg q)$ and by De Morgan's law

$$\neg p \to \neg q \equiv \neg(\neg p \wedge q)$$

16. Construct a truth table to prove a conditional is equivalent to its contra-positive, that is,

$$p \to q \equiv \neg q \to \neg p$$

Answer:

Input		Output		
p	q	$p \to q$	$\equiv$	$\neg q \to \neg p$
0	0	1	1	1
0	1	1	1	1
1	0	0	1	0
1	1	1	1	1

17. Construct a truth table to prove a conditional is not equivalent to its converse, that is,

$$P \to Q \not\equiv Q \to P$$

Answer:

Input		Output		
P	Q	$P \to Q$	$\not\equiv$	$Q \to P$
0	0	1	1	1
0	1	1	0	0
1	0	0	0	1
1	1	1	1	1

18. Construct a truth table to prove a conditional is not equivalent to its inverse, that is,

$$P \to Q \not\equiv \neg P \to \neg Q$$

Answer:

Input		Output		
P	Q	$P \to Q$	$\not\equiv$	$\neg P \to \neg Q$
0	0	1	1	1
0	1	1	0	0
1	0	0	0	1
1	1	1	1	1

19. What is the relationship between the converse and inverse of a conditional $P \to Q$?

Answer: The converse ($Q \to P$) and inverse ($\neg P \to \neg Q$) of $P \to Q$ are equivalent. The inverse of the contra-positive of the converse.

20. Construct a truth table to prove AND distributes over OR, that is,

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

Answer:

P	Q	R	$P\wedge$	$(Q \vee R)$	$\equiv$	$(P \wedge Q)$	$\vee$	$(P \wedge R)$
0	0	0	0	0	1	0	0	0
0	0	1	0	1	1	0	0	0
0	1	0	0	1	1	0	0	0
0	1	1	0	1	1	0	0	0
1	0	0	0	0	1	0	0	0
1	0	1	1	1	1	0	1	1
1	1	0	1	1	1	1	1	0
1	1	1	1	1	1	1	1	1

21. Construct a truth table to prove OR distributes over AND, that is,

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Answer:

p	q	r	$p\vee$	$(q \wedge r)$	$\equiv$	$(p \vee q)$	$\wedge$	$(p \vee r)$
0	0	0	0	0	0	0	0	
0	0	1	0	0	0	0	1	
0	1	0	0	0	1	0	0	
0	1	1	1	1	1	0	1	
1	0	0	1	0	1	1	1	
1	0	1	1	0	1	1	1	
1	1	0	1	0	1	1	1	
1	1	1	1	1	1	1	1	

22. Construct a truth table for the Boolean expression

$$(p \to q) \wedge (\neg p \to r) \wedge (\neg q \wedge \neg r)$$

Answer:

p	q	r	$(p \to q)$	$\wedge$	$(\neg q \to r)$	$\wedge$	$\neg q \wedge \neg r$
0	0	0	1	0	0	0	1
0	0	1	1	1	1	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	0	0
1	0	0	0	0	1	0	1
1	0	1	0	0	1	0	0
1	1	0	1	1	1	0	0
1	1	1	1	1	1	0	0

The Boolean expression is a contradiction. It is the negation of the tautology

$$((p \to q) \wedge (\neg p \to r)) \to (q \vee r)$$

which is named "resolution."

23. Construct a truth table for the Boolean expression

$$((p \wedge q) \to r) \equiv (p \to (q \to r))$$

Is the equivalence `True` or `False`? Explain your answer.

Answer:

P	Q	R	$(P \wedge Q)$	$\rightarrow R$	$\equiv$	$(P \rightarrow$	$(Q \rightarrow R))$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	0
0	1	1	0	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	1	0	1	0	0
1	1	1	1	1	1	1	1

This law is called CURRYING after Haskell Curry. This is a mathematical law that is always `True`.

24. Construct a truth table for the Boolean expressions

$$(P \rightarrow Q) \oplus (\neg P \rightarrow Q)$$

What is a more simple equivalent expression.

Answer:

P	Q	$(P \rightarrow Q)$	$\oplus$	$(\neg P \rightarrow Q)$
0	0	1	1	0
0	1	1	0	1
1	0	0	1	1
1	1	1	0	1

This is equivalent to Q.

25. Identify each statement as a tautology, a contradiction, or a contingency.

(a) $\neg(P \vee \neg Q) = \neg P \wedge Q$

Answer: A tautology, always `True`. Notice it is an application of one of DeMorgan's rules.

(b) $(P \vee (\neg P \wedge Q)) \rightarrow \neg Q$

Answer: If Q is `True`, then $(P \vee (\neg P \wedge Q))$ will be `True` when P is `True` and when it is `False`. On the other hand, if Q is `False`, then then no matter the value of the P the main implication will be `True`. Therefore, the main implication is a contingency. A truth table can also demonstrate the proposition is a contingency

P	Q	$(P \vee (\neg P \wedge Q)$	$\rightarrow$	$\neg Q$
0	0	0	1	1
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

(c) $(Q \vee (\neg P \wedge Q)) \rightarrow \neg P$

Answer: If P is `True`, then $(Q \vee (\neg P \wedge Q))$ will be `True` only when Q is `True`. On the other hand, if P is `False`, then then no matter the value of the Q the main implication will be `True`. Therefore, the main

implication is a contingency. A truth table can also demonstrate the proposition is a contingency

p	q	$(q \vee (\neg p \wedge q)$	$\rightarrow$	$\neg p$
0	0	0	1	1
0	1	1	1	1
1	0	0	1	0
1	1	1	0	0

(d) $(p \rightarrow (q \wedge \neg q)) \rightarrow \neg p$

Answer: A tautology, called reductio ad absurdum. Since $q \wedge \neg q$ is a contracdition, the left-hand side of the main implication can be `True` only if p is `False`. A truth table can also demonstrate the proposition is a tautology.

p	q	$(p$	$\rightarrow$	$(q \wedge q)$	$\rightarrow$	$\neg p$
0	0	0	1	0	1	1
0	1	0	1	0	1	1
1	0	1	0	0	1	0
1	1	1	0	0	1	0

26. Construct a truth table for each of the following Boolean expressions. Identify the expression as a tautology, a contradiction, or a contingency.

(a) $[(p \rightarrow q) \wedge (\neg p \rightarrow r)] \rightarrow (q \vee r)$

Answer:

Input			Output				
p	q	r	$[p \rightarrow q$	$\wedge$	$\neg p \rightarrow r]$	$\rightarrow$	$(q \vee r)$
0	0	0	1	0	0	1	0
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	1	1	0
1	0	1	0	0	1	1	1
1	1	0	1	1	1	1	1
1	1	1	1	1	1	1	1

This is a tautology.

(b) $\neg([p \wedge (p \rightarrow q)] \rightarrow q)$

Answer:

Input		Output					
p	q	$\neg$	$([p$	$\wedge$	$(p \rightarrow q)]$	$\rightarrow$	$q)$
0	0	0	0	0	1	1	0
0	1	0	0	0	1	1	1
1	0	0	1	0	0	1	0
1	1	0	1	1	1	1	1

This is a contradiction.

(c) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Answer:

Input		Output			
P	Q	$\neg$	$(P \wedge Q)$	$\equiv$	$\neg P \vee \neg Q]$
0	0	1	0	1	1
0	1	1	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0

This is a tautology.

(d) $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$

Answer:

Input		Output				
P	Q	$(\neg Q$	$\wedge$	$(P \rightarrow Q))$	$\rightarrow$	$\neg P$
0	0	1	1	1	1	1
0	1	0	0	1	1	1
1	0	1	0	0	1	0
1	1	0	0	1	1	0

This is a tautology.

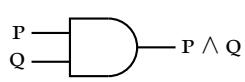
(e) $\neg Q \wedge (\neg P \rightarrow Q) \wedge \neg P$

Answer:

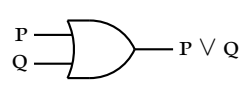
Input		Output				
P	Q	$(\neg Q$	$\wedge$	$(\neg P \rightarrow Q))$	$\wedge$	$\neg P$
0	0	1	0	0	0	1
0	1	0	0	1	0	1
1	0	1	1	1	0	0
1	1	0	0	1	0	0

This is a contradiction.

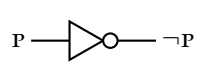
27. Complex computer circuits can be build from three basic gate types, called `And`, `Or`, and `Not`, and drawn as illustrated below.



`And Gate`



`Or Gate`



`Not Gate`

The inputs P and Q are high (1) or low (0) voltages. Fill in the chart below.

P	Q	$\neg$P	P $\wedge$ Q	P $\vee$ Q
0	0	1	0	0
0	1	1	0	1
1	0	0	0	1
1	1	0	1	1

28. How many truth assignments are on n Boolean variables $P_0, P_1 \ldots P_{n-1}$?

    Answer: There are 2^n different truth assignments.

29. Let $\mathbb{B}^n$ be the set of all strings of high or low voltages of length n. For instance there are 4 strings of length 2.

$$\mathbb{B}^2 = \{00, 01, 10, 11\}$$

    What is the cardinality of $\mathbb{B}^n$?

    Answer: There are 2^n bit strings of length n.

30. How many functions can be constructed from $\mathbb{B}^n$ to $\mathbb{B} = \{0, 1\}$ using `And`, `Or`, and `Not` operations?

    Answer: There are 2^n bit strings in $\mathbb{B}^n$. A function maps each bit string to one of 2 Boolean values. That is, there are 2 output choices for each of the 2^n input values. Therefore, there are 2^{2^n} Boolean functions.

31. How many functions can be constructed from $\mathbb{B}^n$ to $\mathbb{B}^m$ using the operations `And`, `Or`, and `Not`?

    Answer: There are 2^n strings in $\mathbb{B}^n$. Each string can be mapped to 2^m Boolean outputs. Therefore, there are $(2^m)^{2^n} = 2^{m2^n}$ functions.

32. A quasi-Boolean function maps $\mathbb{B}^n$ to one of three values: `True`, `False`, or `Don't Care`. How many n-variable quasi-Boolean functions are there?

    Answer: There are 3^{2^n} different quasi-Boolean functions.

33. Using And, Or, and Not operations, construct a Boolean function that behaves as indicated in the truth tables below.

    (a)

Input		Output
P	Q	R
0	0	1
0	1	0
1	0	0
1	1	1

    Answer: One solution is $(\neg P \wedge Q) \vee (P \wedge Q)$. Another solutions is $(\neg P \vee Q) \wedge (P \vee \neg Q)$.

34. It is estimated that there are about 10^{80} atoms in the observable universe. Find n, the number of propositions such that there are more Boolean functions in n variables than there are atoms in the universe.

Answer: There are 2^{2^n} Boolean functions in n variable.

$$\begin{aligned} 2^{2^n} &\geq 10^{80} \quad \text{implies } 2^n \log 2 \geq 80 \\ 2^n \log 2 &\geq 80 \quad \text{implies } 2^n \geq 80/\log 2 \\ 80/\log 2 &\approx 266 \quad \text{implies } n \geq 9 \end{aligned}$$

Problems on Predicate Logic

1. For what phrase does the symbol $\forall$ stand?
   Answer: $\forall$ is shorthand for "for all" or "for each" or "for every".
2. For what phrase does the symbol $\exists$ stand?
   Answer: $\exists$ is shorthand for "there exists" or "for at least one" or "for some".
3. Use mathematical notation to write the following predicates.
   (a) Every natural number greater than 0 lies within an interval bounded by powers of 2.
   Answer:
   $$(\forall n \in \mathbb{N},\, n > 0)(\exists m \in \mathbb{N})(2^{m-1} \leq n < 2^m)$$
   (b) There is no largest prime number.
   Answer: You can say that for every natural number n, there is a prime number p that is larger than n.
   $$(\forall n \in \mathbb{N})(\exists p \in \mathbb{P})(p > n)$$
   (c) Every integer a can be divided by a non-zero integer n to produce an integer quotient and remainder.
   Answer:
   $$(\forall a,\, n \in \mathbb{Z},\, n \neq 0)(\exists q,\, r \in \mathbb{Z})(a = qn + r)$$
4. Use mathematical notation to write the predicate "The weak can never forgive," that is attributed to Mahatma Gandhi,
   Answer: Let the domain be $\mathbb{P}$, the set of all people, and let $p \in \mathbb{P}$ represent a person. Let $\mathrm{W}(p)$ be the predicate "p is weak" and let $\mathrm{F}(p)$ be the predicate "p can forgive." Gandhi's statement is $\mathrm{W}(p) \rightarrow \neg\mathrm{F}(p)$ (If person P is weak, then P cannot forgive).
5. The predicate statements below come from (Carroll, 1958) and other sources. Write them using mathematical notation.
   (a) No Frenchman likes plumpudding.
   Answer: $(\forall p \in \mathbb{P})(\mathrm{F}(p) \rightarrow \neg\mathrm{LP}(p))$, where $\mathrm{F}(p)$ is the predicate "p is a Frenchman" and $\mathrm{LP}(p)$ is the predicate "p likes plumpudding. Equivalent statements are $(\forall p \in \mathbb{P})(\mathrm{LP}(p) \rightarrow \neg\mathrm{F}(p))$ and $\neg(\exists p \in \mathbb{P})(\mathrm{F}(p) \wedge \mathrm{LP}(p))$.

(b) All Englishmen like plumpudding.
Answer: $(\forall p \in \mathbb{P})(\mathrm{E}(p) \to \mathrm{LP}(p))$.

(c) Some thin persons are not cheerful.
Answer: $(\exists p \in \mathbb{P})(\textsc{Thin}(p) \wedge \neg\textsc{Cheerful}(p))$.

(d) All pigs are fat.
Answer: $(\forall p \in \mathbb{PIG})(\textsc{Fat}(p))$.

(e) Some lessons are difficult.
Answer: $(\exists l \in \mathbb{L})(\mathrm{D}(l))$.

(f) All clever people are popular.
Answer: $(\forall p \in \mathbb{P})(\textsc{Clever}(p) \to \textsc{Popular}(p))$.

(g) Some healthy people are fat.
Answer: $(\exists p \in \mathbb{P})(\mathrm{H}(p) \wedge \textsc{Fat}(p))$.

(h) Some unauthorized reports are false;
Answer: Let $\textsc{u}(x)$ be the predicate: x is an unauthorized report, and let $f(x)$ be the predicate: x is an false report.

$$(\exists x)(u(x) \wedge f(x))$$

All authorized reports are trustworthy.
Answer: Let $\textsc{t}(x)$ be the predicate: x is a trustworthy report.

$$(\forall x)(\neg u(x) \to t(x))$$

Some false reports are not trustworthy.
Answer:

$$(\exists x)(f(x) \wedge \neg t(x))$$

(i) Babies are illogical;
Answer: Let $\textsc{b}(x)$ be the predicate: x is a baby, and let $\textsc{l}(x)$ be the predicate: x is logical.

$$(\forall x)(\textsc{b}(x) \to \neg\textsc{l}(x))$$

nobody is despised who can manage a crocodile;
Answer: Let $\textsc{c}(x)$ be the predicate: x is able to manage a crocodile, and let $\textsc{d}(x)$ be the predicate: x is despised.

$$\neg(\exists x)(\textsc{d}(x) \wedge \textsc{c}(x))$$

or

$$(\forall x)(\neg\textsc{d}(x) \vee \neg\textsc{c}(x))$$

illogical persons are despised.
Answer:

$$(\forall x)(\neg\textsc{l}(x) \to \textsc{d}(x))$$

Babies cannot manage a crocodile.
Answer:

$$(\forall x)(\textsc{b}(x) \to \neg\textsc{c}(x))$$

(j) No sinner is ever saved after the first twenty minutes of a sermon. (Mark Twain)
Answer: Let $\mathrm{s}(x)$ be the predicate: x is a sinner. Let $\mathrm{v}(x)$ be the predicate: x has been saved. Let $\mathrm{R}(x)$ be the predicate: The sermon has lasted x or more minutes.

$$(\forall x)(\mathrm{R}(20) \to (\mathrm{s}(x) \to \neg \mathrm{v}(x)))$$

(k) You do not really understand something unless you can explain it to your grandmother. (Albert Einstein)

6. Write the following statements in predicate notation.

(a) For every x in $\mathbb{X}$ there is a y in $\mathbb{Y}$ such that $f(x) = y$.
Answer:

$$(\forall x \in \mathbb{X})(\exists y \in \mathbb{Y})(f(x) = y)$$

This is a statement that f is a (total) function.

(b) For every y in $\mathbb{Y}$ there is a x in $\mathbb{X}$ such that $f(x) = y$.
Answer:

$$(\forall y \in \mathbb{Y})(\exists x \in \mathbb{X})(f(x) = y)$$

This is a statement that f is a onto its co-domain.

(c) For every x_1, x_2 in $\mathbb{X}$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
Answer:

$$(\forall x_1, x_2 \in \mathbb{X})(f(x_1) = f(x_2) \to x_1 = x_2$$

This is a statement that f one-to-one. (Assume f is a function.)

7. Which of the statements below are `True` and which are `False`?

(a) $(\forall n \in \mathbb{N})(n^2 > 0)$
Answer: This is `False`; it fails to hold for $0 \in \mathbb{N}$.

(b) $(\exists n \in \mathbb{N})(n^2 > 0)$
Answer: This is `True`; it holds for $n = 1$.

(c) $(\forall n \in \mathbb{N})(n^2 \leq 0)$
Answer: This is `False`; it fails to hold for $1 \in \mathbb{N}$.

(d) $(\exists n \in \mathbb{N})(n^2 \leq 0)$
Answer: This is `True`; it holds for $n = 0$.

(e) $(\forall n \in \mathbb{N})(\exists k \in \mathbb{N})(n = 4k \vee n = 4k+1 \vee n = 4k+2 \vee n = 4k+3)$
Answer: This is `True`; every natural number n has a remainder of 0, 1, 2 or 3 when divided by 4.

(f) $(\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(n = 4k \vee n = 4k+1 \vee n = 4k+2 \vee n = 4k+3)$
Answer: This is `False`; there is not a single natural number k such that every natural number is equal to $4k$, $4k+1$, $4k+2$ or $4k+3$.

(g) $(\forall q \in \mathbb{Q})(q \in \mathbb{N})$
Answer: This is `False`; there are rational numbers, such as $q = 0.5$, that are not natural numbers.

(h) $(\forall n \in \mathbb{N})(n \in \mathbb{Q})$
Answer: This is `True`; every natural number $n = n/1$ is a rational number.

8. Let $\mathrm{P}(x)$ be the statement $x^2 = x + 1$ over the domain of real numbers. What is the truth value of the following statements?

(a) $\text{P}(0)$
Answer: $\text{P}(0)$ is `False`.

(b) $\text{P}((1+\sqrt{5})/2)$
Answer: $\text{P}(1+\sqrt{5})/2)$ is `True`.

(c) $(\forall x \in \mathbb{R})(\text{P}(x))$
Answer: $(\forall x \in \mathbb{R})(\text{P}(x))$ is `False`.

(d) $(\forall x \in \mathbb{R})(\neg\text{P}(x))$
Answer: $(\forall x \in \mathbb{R})(\neg\text{P}(x))$ is `False`.

(e) $(\exists x \in \mathbb{R})(\text{P}(x))$
Answer: $(\exists x \in \mathbb{R})(\text{P}(x))$ is `True`.

(f) $(\exists x \in \mathbb{R})(\neg\text{P}(x))$
Answer: $(\exists x \in \mathbb{R})(\neg\text{P}(x))$ is `True`.

9. This question is from (Papadimitriou, 1994). Let $\textsc{canfool}(p, t)$ be the proposition "You can fool person p at time t." Write Abraham Lincoln's famous quotation

   You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time.

   using the quantifiers over people p and times t and the $\textsc{canfool}(p, t)$ predicate.
   Answer: $(\exists p)(\forall t)(\textsc{canfool}(p, t)) \wedge (\forall p)(\exists t)(\textsc{canfool}(p, t)) \wedge \neg(\forall p)(\forall t)(\textsc{canfool}(p, t))$.

10. This question is from the MIT OpenCourseWare course on Discrete Mathematics. Suppose $S(n)$ is a predicate on natural numbers, n, and suppose

$$(\forall k \in \mathbb{N})(S(k) \to S(k+2)) \tag{1}$$

holds `True`, Are the statements below always `True`, always `False`, or sometimes `True` and sometimes `False`?

- Consider the statement
$$(\forall n \leq 100)(S(n)) \wedge (\forall n > 100)(\neg S(n))$$
  Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
  Answer: In this case, S is `True` for n up to 100 and `False` from 101 on. So $S(99)$ is `True`, but $S(101)$ is `False`. That means that $\neg(S(k) \to S(k+2))$ for $k = 99$. This can not be `True` if equation 1 is `True`.
- Consider the statement
$$S(1) \to (\forall n)(S(2n+1))$$
  Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
  Answer: This assertion says that if $S(1)$ holds, then $S(n)$ holds for all odd n. This case is always `True`.
- Consider the statement
$$(\exists n)(S(2n) \to (\forall n)(S(2n+2)))$$
  Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
  Answer: If $S(n)$ is always `True`, this assertion holds. So this case is possible. If $S(n)$ is `True` only for even integers greater than 4, (1) holds, but this assertion is `False`. So this case does not always hold.
- Consider the statement
$$(\exists n)(\exists m > n)(S(2n) \wedge \neg S(2m))$$
  Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
  Answer: This assertion says that S holds for some even number, $2n$, but not for some other larger even number, $2m$. However, if $S(2n)$ holds, we can apply n — m times to conclude $S(2m)$ also holds. This case is impossible.

- Consider the statement

$$(\exists n)(S(n)) \rightarrow (\forall n)(\exists m > n)(S(m))$$

Is it always `True`, always `False`, or sometimes `True` and sometimes `False`?
Answer: This assertion says that if S holds for some n, then for every number, there is a larger number, m, for which S also holds. Since (1) implies that if there is one n for which $S(n)$ holds, there are an infinite, increasing chain of k's for which $S(k)$ holds, this case is always `True`.

Problems on Sets

1. (Be able to identify the symbols used in discussions about sets.) Match the term with its symbol.

(a) (g) INTERSECT	(a) $\cup$
(b) (c) Set Equality	(b) $\ominus$
(c) (e) Set Complement	(c) $=$
(d) (b) Symmetric Difference	(d) $\subset$
(e) (j) Universal Set	(e) $\overline{}$
(f) (f) Empty Set	(f) $\varnothing$
(g) (a) UNION	(g) $\cap$
(h) (h) Subset	(h) $\subseteq$
(i) (d) Proper Subset	(i) $\in$
(j) (i) Member (Element) of	(j) $\mathbb{U}$

2. (Know the names and elements of sets that commonly occur in computing.) Describe these sets.

(a) $\mathbb{B}$
Answer: $\mathbb{B}$ is the set of bits

$$\mathbb{B} = \{0, 1\}$$

(b) $\mathbb{O}$
Answer: $\mathbb{O}$ is the set of octal numerals

$$\mathbb{O} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

(c) $\mathbb{D}$
Answer: $\mathbb{D}$ is the set of digits

$$\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

(d) $\mathbb{H}$
Answer: $\mathbb{H}$ is the set of hexadecimal numerals

$$\mathbb{H} = \{0, 1, \ldots, 9, A, B, \ldots, F\}$$

(e) $\mathbb{N}$
Answer: $\mathbb{N}$ is the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

Some authors do not include 0

(f) $\mathbb{P}$
Answer: $\mathbb{P}$ is the set of prime numbers

$$\mathbb{P} = \{2, 3, 5, 7, \ldots\}$$

(g) $\mathbb{Z}$
Answer: $\mathbb{Z}$ is the set of integers

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

(h) $\mathbb{Z}_n$
Answer: $\mathbb{Z}_n$ is the set of integers modulo n

$$\mathbb{Z}_n = \{0, 1, 2, 3, \ldots, (n-1)\}$$

(i) $\mathbb{Z}^+$
Answer: $\mathbb{Z}^+$ is the set of positive integers

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

(j) $\mathbb{Q}$
Answer: $\mathbb{Q}$ is the set of rationals

$$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$$

(k) $\mathbb{R}$
Answer: $\mathbb{R}$ is the set of real numbers.

(l) $\emptyset$
Answer: $\emptyset$ is the empty set; the set with no elements. Some people write $\{\}$ for the empty set.

3. (Write sets in mathematical notation.) Describe the following sets. List representative patterns for the elements in the sets. Give functions that enumerate the values in the sets.

(a) The set of even integers, denoted $[0]_2$.
Answer: The set of even integers is

$$\{\ldots, -4, -2, 0, 2, 4, \ldots\} = \{2n : n \in \mathbb{Z}\}$$

A name for the even integers is $[0]_2$, the set of integers congruent to 0 mod 2. Even integers can be written as $2n$ where n is an integer. That is, the function $e(n) = 2n$ maps the integers to the even integers.

(b) The set of odd integers, denoted $[1]_2$.
Answer: The set of odd integers is

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\} = \{2n + 1 : n \in \mathbb{Z}\}$$

A common name for the odd integers $[1]_2$, the set of integers congruent to 1 mod 2. Odd integers can be written as $2n + 1$ where n is an integer. That is, the function $o(n) = 2n + 1$ maps the integers to the odd integers.

(c) The set of integers that have a remainder of 2 when divided by 3, denoted $[2]_3$.
Answer: The remainder 2 on divide by 3 set is

$$\{\ldots, -7, -4, -1, 2, 5, 8, \ldots\} = \{3n + 2 : n \in \mathbb{Z}\}$$

A common name for the odd integers $[2]_3$, the set of integers congruent to 2 mod 3. The numbers congruent to 2 mod 3 can be written as $3n + 2$ where n is an integer. That is, the function $f(n) = 3n + 2$ maps the integers to the integers congruent to 2 mod 3.

(d) The set of Mersenne numbers, denoted $\mathbb{M}$.
Answer: The Mersenne numbers are one less than a power of 2.

$$\mathbb{M} = \{0, 1, 3, 7, 15, \ldots\} = \{2^n - 1 : n \in \mathbb{N}\}$$

The Mersenne numbers can be written as $2^n - 1$ where n is a natural number. That is, the function $M(n) = 2^n - 1$ maps the natural numbers to the Mersenne numbers. Notice the domain of the function $M(n)$ could be extended to the integers.

$$M(-1) = 2^{-1} - 1 = -\frac{1}{2},\ M(-2) = 2^{-2} - 1 = -\frac{3}{4},\ M(-3) = 2^{-3} - 1 = -\frac{7}{8}$$

In general,

$$\begin{aligned} M(-n) &= 2^{-n} - 1 \\ &= \frac{1 - 2^n}{2^n} \\ &= -\frac{M(n)}{2^n} \end{aligned}$$

(e) The set of Triangular numbers, denoted $\mathbb{T}$.

Answer: The triangular numbers are the partial sums of natural numbers.

$$\mathbb{T} = \{0, 1, 3, 6, 10, \ldots\} = \left\{ \frac{n(n-1)}{2} : n \in \mathbb{N} \right\}$$

The triangular numbers are the binomial coefficients, they are column 1 in Pascal's triangle, and can be written as

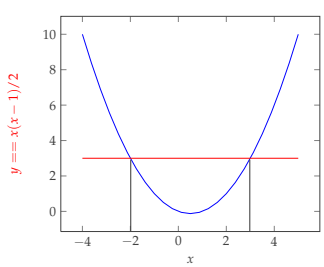
$$\binom{n}{2} = \frac{n(n-1)}{2},\ n \geq 0$$

That is, the function $T(n) = n(n-1)/2$ maps the natural numbers to the triangular numbers. Notice the domain of the function $T(n)$ could be extended to the integers.

$$T(-1) = -1(-2)/2 = 1,\ T(-2) = -2(-3)/2 = 3,\ T(-3) = -3(-4)/2 = 6$$

In general, for $n > 0$

$$\begin{aligned} T(-n) &= \frac{-n(-n-1)}{2} \\ &= \frac{n(n+1)}{2} \\ &= T(n+1) \end{aligned}$$



(f) The set of Fibonacci numbers, denoted $\mathbb{F}$.
Answer: The Fibonacci numbers are

$$\mathbb{F} = \{0, 1, 1, 2, 3, 5, 8, \ldots\} = \{F_n : F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1\}$$

4. (Describe sets defined by algebraic constraints.) Describe the following sets.

(a) The set of solutions to the polynomial equation $x^2 - x - 1 = 0$.
Answer: The set has two members: the golden ratio $\varphi = (1 + \sqrt{5})/2$ and its conjugate $\overline{\varphi} = (1 - \sqrt{5})/2$. That is the solution set has two members

$$\left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\}$$

(b) The set of solutions to the equation $|x| - 1 = 0$.
Answer: The set has two members: 1 and -1.

$$\{1, -1\}$$

(c) The set of solutions to the inequality $|3x - 2| \leq 5$.
Answer: There are two inequalities:

$$3x - 2 \leq 5 \quad \text{and} \quad -3x + 2 \leq 5$$

which has an interval of solutions

$$\left[-1 \leq x \leq \frac{7}{3}\right] = \left\{x : -1 \leq x \leq \frac{7}{3}\right\}$$

(d) The set of solutions to the inequality $|3x - 2| > 5$.
Answer: There are two inequalities:

$$3x - 2 > 5 \quad \text{and} \quad 3x - 2 < -5$$

which has a union of intervals as solutions

$$\left(\frac{7}{3}, \infty\right) \cup (-\infty, -1)$$

5. (Know the meaning to common phrase and write them in mathematical notation.) What notation would be used to stand for the following phrases?

(a) x is an element of $\mathbb{X}$.
Answer: $x \in \mathbb{X}$.

(b) x is not an element of $\mathbb{X}$.
Answer: $x \notin \mathbb{X}$.

(c) $\mathbb{X}$ is a subset of $\mathbb{Y}$.
Answer: $\mathbb{X} \subseteq \mathbb{Y}$.

(d) $\mathbb{X}$ is a proper subset of $\mathbb{Y}$.
Answer: $\mathbb{X} \subset \mathbb{Y}$.

(e) $\mathbb{X}$ is not a subset of $\mathbb{Y}$.
Answer: $\mathbb{X} \not\subseteq \mathbb{X}$.

(f) The union of $\mathbb{X}$ and $\mathbb{Y}$.
Answer: $\mathbb{X} \cup \mathbb{Y}$.

(g) The intersection of $\mathbb{X}$ and $\mathbb{Y}$.
Answer: $\mathbb{X} \cap \mathbb{Y}$.

(h) The complement of $\mathbb{X}$.
Answer: $\overline{\mathbb{X}}$.

(i) Elements in $\mathbb{X}$ but not in $\mathbb{Y}$.
Answer: $\mathbb{X} - \mathbb{Y} = \mathbb{X} \cap \overline{\mathbb{Y}}$.

(j) The empty set
Answer: $\emptyset$, although some use $\{\}$.

(k) The sets $\mathbb{X}$ and $\mathbb{Y}$ are disjoint
Answer: $\mathbb{X} \cap \mathbb{Y} = \emptyset$.

(l) The sets $\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_{n-1}$ are mutually disjoint
Answer: $(\forall i, j, i \neq j)(\mathbb{X}_i \cap \mathbb{X}_j = \emptyset)$.

6. (Evaluate decision problems about set membership and subset relations.) Answer `True` or `False`.

(a) $2 \in \{\{2\}\}$.
Answer: This statement is `False`.

(b) $\{2\} \in \{\{2\}\}$.
Answer: This statement is `True`.

(c) $2 \in \{2\}$.
Answer: This statement is `True`.

(d) $2 \notin \{2\}$.
Answer: This statement is `False`.

(e) $\emptyset \in \emptyset$.
Answer: This statement is `False`.

(f) $\emptyset \in \{\emptyset\}$.
Answer: This statement is `True`.

(g) $0 \in \emptyset$.
Answer: This statement is `False`.

(h) $\emptyset \subseteq \{x\}$.
Answer: This statement is `True`.

(i) $\emptyset = \{\emptyset\}$.
Answer: This statement is `False`.

(j) $\{x\} \subseteq \{x\}$.
Answer: This statement is `True`.

(k) $\{x\} \in \{x\}$.
Answer: This statement is `False`.

(l) $\{x\} \in \{\{x\}\}$.
Answer: This statement is `True`.

(m) $\{x\} \subseteq \{\{x\}\}$.
Answer: This statement is `False`.

(n) $\{(0,1)\} \subseteq \{(0,0),(0,1),(1,0),(1,1)\}$
Answer: This statement is `True`.

(o) $(0,1) \subseteq \{(0,0),(0,1),(1,0),(1,1)\}$
Answer: This statement is `False`.

(p) $\{(0,1)\} \in \{(0,0),(0,1),(1,0),(1,1)\}$
Answer: This statement is `True`.

(q) $(0,1) \in \{(0,0),(0,1),(1,0),(1,1)\}$
Answer: This statement is `True`.

7. Is $x \in \emptyset$ ever `True`?
Answer: No, there is no element in the empty set.

8. Is $\emptyset \subseteq \mathbb{X}$ ever `True`?
Answer: Yes, the empty set is always a subset of any set $\mathbb{X}$. This is because the conditional $x \in \emptyset \rightarrow x \in \mathbb{X}$ is always `True`, because the assumption $x \in \emptyset$ is always `False`.

9. (Evaluate decision problems about set operations.) Answer `True` or `False`.

(a) $\mathbb{X} \cup \emptyset = \mathbb{X}$
Answer: This is `True`. The empty set is the identity element for the Union operation.

(b) $\mathbb{X} \cap \mathbb{U} = \mathbb{X}$
Answer: This is `True`. The universal set is the identity element for the Intersection operation.

(c) $\mathbb{X} \cup \overline{\mathbb{X}} = \mathbb{U}$
Answer: This is `True`. This is the complement

law for the Union operation.

(d) $\mathbb{X} \cap \overline{\mathbb{X}} = \emptyset$
Answer: This is `True`. This is the complement law for the Intersection operation.

(e) $(\mathbb{X} \cup \mathbb{Y}) \cup \mathbb{Z} = \mathbb{X} \cup (\mathbb{Y} \cup \mathbb{Z})$
Answer: This is `True`. This is the associative law for the Union operation.

(f) $(\mathbb{X} \cap \mathbb{Y}) \cap \mathbb{Z} = \mathbb{X} \cap (\mathbb{Y} \cap \mathbb{Z})$
Answer: This is `True`. This is the associative law for the Intersection operation.

(g) $\mathbb{X} \cup \mathbb{Y} = \mathbb{Y} \cup \mathbb{X}$
Answer: This is `True`. This is the commutative law for the Union operation.

(h) $\mathbb{X} \cap \mathbb{Y} = \mathbb{Y} \cap \mathbb{X}$
Answer: This is `True`. This is the commutative law for the Intersection operation.

(i) $\mathbb{X} \cup (\mathbb{Y} \cap \mathbb{Z}) = (\mathbb{X} \cup \mathbb{Y}) \cap (\mathbb{X} \cup \mathbb{Z})$
Answer: This is `True`. This is the distributive law for the Union over Intersection.

(j) $\mathbb{X} \cap (\mathbb{Y} \cup \mathbb{Z}) = (\mathbb{X} \cap \mathbb{Y}) \cup (\mathbb{X} \cap \mathbb{Z})$
Answer: This is `True`. This is the distributive law for the Intersection over Union.

10. (Know the set difference operator.)

(a) Write the difference $\mathbb{Y} - \mathbb{X}$ using the standard union, intersection, and set complement operators.
Answer: $\mathbb{Y} - \mathbb{X} = \mathbb{Y} \cap \overline{\mathbb{X}}$.

(b) Is $\mathbb{X} - (\mathbb{Y} - \mathbb{V}) = (\mathbb{X} - \mathbb{Y}) - \mathbb{V}$
Answer: No, as a counterexample consider the sets $\mathbb{X} = \{0, 1\}$, $\mathbb{Y} = \{0\}$, and $\mathbb{V} = \{1\}$. Then

$$\mathbb{X} - (\mathbb{Y} - \mathbb{V}) = \{0, 1\} - (\{0\} - \{1\}) = \{1\}$$

and

$$(\mathbb{X} - \mathbb{Y}) - \mathbb{V} = (\{0, 1\} - \{0\}) - \{1\} = \emptyset$$

11. (Evaluate decision problems about set equality.) Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, be subsets of a universal set $\mathbb{U}$. Identify each statement as either a tautology, a contradiction, or a contingency.

(a) $\mathbb{A} \cap \emptyset = \mathbb{U}$
Answer: A contingency. Only `True` when $\mathbb{U} = \emptyset$.

(b) $\mathbb{A} \cap (\mathbb{A} \cup \mathbb{B}) = \mathbb{A}$
Answer: A tautology. Known as an absorption rule.

Input		
$z \in \mathbb{A}$	$z \in \mathbb{B}$	$z \in \mathbb{A} \cap (\mathbb{A} \cup \mathbb{B})$
0	0	0
0	1	0
1	0	1
1	1	1

(c) $\mathbb{A} \cap (\mathbb{B} \cup \mathbb{C}) = (\mathbb{A} \cup \mathbb{B}) \cup (\mathbb{A} \cap \mathbb{C})$
Answer: This is a contingency. Construct a truth table to compute when the proposition is `True` and

when it is `False`.

Input				
$z \in \mathbb{A}$	$z \in \mathbb{B}$	$z \in \mathbb{C}$	$z \in \mathbb{A} \cap (\mathbb{B} \cup \mathbb{C})$	$z \in \mathbb{A} \cap (\mathbb{B} \cup \mathbb{C})$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

The third row of the table shows that that sets where there is an element $z \in \mathbb{B}$ that is not in either $\mathbb{A}$ or $\mathbb{C}$ makes $z \notin \mathbb{A} \cap (\mathbb{B} \cup \mathbb{C})$ and $z \in \mathbb{A} \cap (\mathbb{B} \cup \mathbb{C})$, which shows the two set expressions are not necessarily equal On the other hand, the seventh row of the table shows when $\mathbb{A} = \mathbb{B}$, then the two expressions are identical.

12. (Be able to count the number of elements in a finite set.) What is the cardinality of each of these sets?

(a) $\emptyset$
Answer: The cardinality of the empty set is 0.

(b) $\{\emptyset\}$
Answer: The cardinality of the set containing only the empty set is 1.

(c) $\{\emptyset, \{\emptyset\}\}$
Answer: The cardinality of the set containing the empty set and the set containing the empty set is 2.

(d) $\mathbb{D}$
Answer: The cardinality of the digits is 10.

(e) $\{x : x^2 - x - 1 = 0\}$.
Answer: The cardinality of this set is 2.

13. (Know there is a hierarchy of infinite sets.)

(a) $\mathbb{N}$
Answer: The cardinality of the natural numbers is $\aleph_0$.

(b) $[1]_2$
Answer: The cardinality of the odd integers is $\aleph_0$.

(c) $\mathbb{Z}$
Answer: The cardinality of the integers is $\aleph_0$.

(d) $\mathbb{Q}$
Answer: The cardinality of the rational numbers is $\aleph_0$.

(e) $\mathbb{P}$
Answer: The cardinality of the prime numbers is $\aleph_0$.

(f) $\mathbb{R}$
Answer: The cardinality of the real numbers is $\aleph_1$.

(g) The closed interval $[0, 1]$
Answer: The cardinality of the closed interval $[0, 1]$ is $\aleph_1$.

(h) The Cantor set.
Answer: This is a set that serves as an important example in areas of advanced mathematics. If it constructed from recurively removing the inner third of the intervals that exist starting from the interval $[0, 1]$. This is beyond the scope of this course, but well worth studying.

14. (Know the set of all subsets (power set) of a set.) What is the power set of each of these sets?

(a) $\{0\}$
Answer: The power set of $\{0\}$ is $2^{\{0\}} = \{\emptyset, \{0\}\}$.

(b) $\left\{x : x^2 - x - 1 = 0\right\}$
Answer: The power set of this set is $\left\{\emptyset, \left\{(1+\sqrt{5})/2\right\}, \left\{(1-\sqrt{5})/2\right\}, \left\{(1+\sqrt{5})/2, (1-\sqrt{5})/2\right\}\right\}$.

(c) $\emptyset$
Answer: The power set of the empty set is $2^{\emptyset} = \{\emptyset\}$.

(d) $\{\emptyset\}$
Answer: The power of the set containing only the empty set is $2^{\{\emptyset\}} = \{\emptyset, \{\emptyset\}\}$.

15. (Know the functional relationship between the cardinality of a set and the cardinality of its power set.) What is the cardinality of each of these sets?

(a) The power set of $\{0\}$.
Answer: The cardinality of the power set of $\{0\}$ is 2.

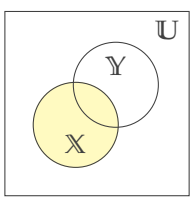
(b) The power set of $\left\{x : x^2 - x - 1 = 0\right\}$.
Answer: The cardinality of the power set of $\left\{x : x^2 - x - 1 = 0\right\}$ is 4.

(c) The power set of $\mathbb{D}$, the set of digits.
Answer: The cardinality of the power set of $\mathbb{D}$ is $2^{10} = 1024$.

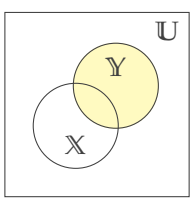
(d) The power set of $\mathbb{H}$, the set of hexadecimal numerals.
Answer: The cardinality of the power set of $\mathbb{H}$ is $2^{16} = 65536$.

16. (Describe sets visually.) Shade the Venn diagram to indicate the given set is <u>not</u> empty.

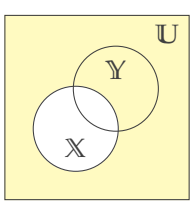
(a) $\mathbb{X}$.
Answer:



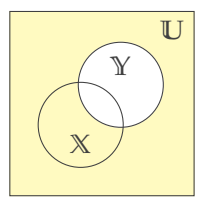
(b) $\mathbb{Y}$.
Answer:



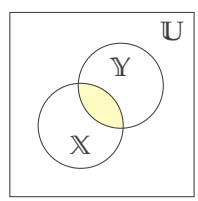
(c) $\overline{\mathbb{X}}$.
Answer:



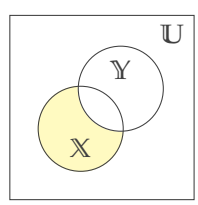
(d) $\overline{\mathbb{Y}}$.
Answer:



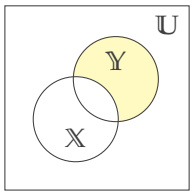
(e) $\mathbb{X} \cap \mathbb{Y}$.
Answer:



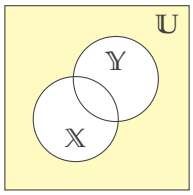
(f) $\mathbb{X} \cap \overline{\mathbb{Y}}$.
Answer:



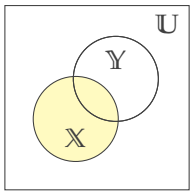
(g) $\overline{\mathbb{X}} \cap \mathbb{Y}$.
Answer:



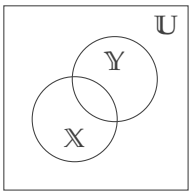
(h) $\overline{\mathbb{X}} \cap \overline{\mathbb{Y}}$.
Answer:



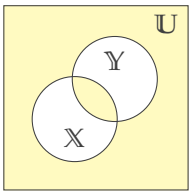
(i) $(\mathbb{X} \cap \mathbb{Y}) \cup (\mathbb{X} \cap \overline{\mathbb{Y}})$.
Answer:



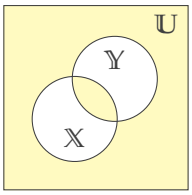
(j) $(\mathbb{X} \cap \mathbb{Y}) \cap (\overline{\mathbb{X}} \cap \overline{\mathbb{Y}})$.
Answer:



(k) $(\mathbb{X} \cap \mathbb{Y}) \cup (\overline{\mathbb{X}} \cap \overline{\mathbb{Y}})$.
Answer:



(l) $(\mathbb{X} \cap \mathbb{Y}) \cup (\overline{\mathbb{X}} \cap \overline{\mathbb{Y}})$.
Answer:

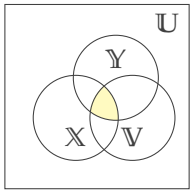


17. (Be able to count the maximum number of regions defined by two sets within a universe.) How many different regions can be shaded from two intersecting circles inside a rectangle? (Include not shading any region at all in your count.)

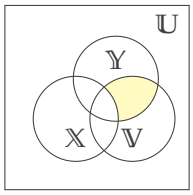
Answer: There are $2^2 = 4$ disjoint regions. There are $2^{2^2} = 2^4 = 16$ different regions.

18. (Describe sets visually.) Shade the Venn diagram to indicate the given region is NOT empty.

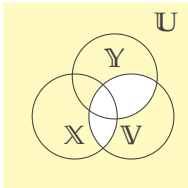
(a) $\mathbb{X} \cap \mathbb{Y} \cap \mathbb{V}$.
Answer:



(b) $\overline{\mathbb{X}} \cap \mathbb{Y} \cap \mathbb{V}$.
Answer:



(c) $\overline{\mathbb{X}} \cap \overline{\mathbb{Y}} \cup \overline{\mathbb{V}}$.
Answer:



(d) $\overline{\mathbb{X}} \cap \mathbb{Y} \cap \mathbb{V}$.

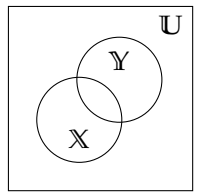
(e) $\overline{\mathbb{X}} \cap \mathbb{Y} \cap \mathbb{V}$.

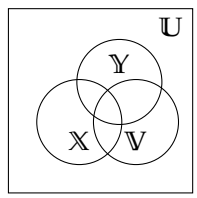
(f) $\overline{\mathbb{X}} \cap \overline{\mathbb{Y}} \cup \overline{\mathbb{V}}$.

19. (Be able to count the maximum number of regions defined by three sets within a universe.) How many different regions can be shaded from three intersecting circles inside a rectangle? (Include not shading any region at all in your count.)

Answer: There are $2^3 = 8$ disjoint regions. There are $2^{2^3} = 2^8 = 256$ different regions.

20. (Be able to develop a sequence that counts simple crossings in Venn diagrams.) A CROSSING is an intersection of two curves. A Venn diagram with 2 circles has 2 crossings while a Venn diagram with 3 circles has 6 crossings.





(a) Draw four ellipses with 14 crossings.

(b) How many crossing does a Venn diagram with n closed curves have?

Answer: Based on the first few cases, make the hypothesis that a Venn diagram with n closed curves has

$$2 + 4 + 8 + 16 + \cdots + 2^{n-1} = 2^n - 2$$

crossings.

21. (Evaluate decision problems about set equality.) Let $\mathbb{X}$ and $\mathbb{Y}$ be subsets of some universal set $\mathbb{U}$. Are the following statements `True` or `False`?

(a) $\overline{\overline{X}} = \overline{X}$

Answer: `False`

(b) $\mathbb{X} \cup \emptyset = \mathbb{X}$

Answer: `True`

(c) $\mathbb{X} \cap \mathbb{U} = \mathbb{X}$

Answer: `True`

(d) $\mathbb{X} \cap \overline{\mathbb{U}} = \mathbb{U}$

Answer: `False`

(e) $\mathbb{X} \cap \overline{\mathbb{U}} = \emptyset$

Answer: `True`

(f) $\overline{\mathbb{X} \cap \mathbb{Y}} = \overline{\mathbb{X}} \cup \overline{\mathbb{Y}}$

Answer: `True`

(g) $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{X}$

Answer: `True`

22. (Know the sets determined by common sequences and be able to compute set operations on them.) Let

$$\mathbb{M}_{10} = \{0, 1, 3, 7, 15, 31, 63, 127, 255, 511\}$$

be the set of the first 10 Mersenne numbers. Let

$$\mathbb{T}_{10} = \{0, 1, 3, 6, 10, 15, 21, 28, 36, 45\}$$

be the set of the first 10 (distinct) triangular numbers. Let

$$\mathbb{F}_{10} = \{0, 1, 2, 3, 5, 8, 13, 21, 34, 55\}$$

be the set of the first 10 (distinct) Fibonacci numbers. Find

(a) $\mathbb{M}_{10} \cup \mathbb{T}_{10}$
Answer:

$$\mathbb{M}_{10} \cup \mathbb{T}_{10} = \{0, 1, 3, 6, 7, 10, 15, 21, 28, 31, 36, 45, 63, 127, 255, 511\}$$

(b) $\mathbb{M}_{10} \cap \mathbb{T}_{10}$
Answer:

$$\mathbb{M}_{10} \cap \mathbb{T}_{10} = \{0, 1, 3, 15\}$$

(c) $\mathbb{M}_{10} - \mathbb{T}_{10}$
Answer:

$$\mathbb{M}_{10} - \mathbb{T}_{10} = \{7, 31, 63, 127, 511\}$$

(d) $\mathbb{T}_{10} - \mathbb{M}_{10}$
Answer:

$$\mathbb{T}_{10} - \mathbb{M}_{10} = \{6, 10, 21, 28, 36, 45\}$$

(e) $(\mathbb{M}_{10} \cup \mathbb{T}_{10}) \cap \mathbb{F}_{10}$
Answer:

$$(\mathbb{M}_{10} \cup \mathbb{T}_{10}) \cap \mathbb{F}_{10} = \{0, 1, 3, 21\}$$

(f) $\mathbb{M}_{10} \cup (\mathbb{T}_{10} \cap \mathbb{F}_{10})$
Answer:

$$\mathbb{M}_{10} \cup (\mathbb{T}_{10} \cap \mathbb{F}_{10}) = \{0, 1, 3, 7, 15, 21, 31, 63, 127, 255, 511\}$$

(g) $(\mathbb{M}_{10} - \mathbb{T}_{10}) - \mathbb{F}_{10}$
Answer:

$$(\mathbb{M}_{10} - \mathbb{T}_{10}) - \mathbb{F}_{10} = \{7, 31, 63, 127, 511\}$$

(h) $\mathbb{M}_{10} - (\mathbb{T}_{10} - \mathbb{F}_{10})$
Answer:

$$\mathbb{M}_{10} - (\mathbb{T}_{10} - \mathbb{F}_{10}) = \{7, 31, 63, 127, 511\}$$

23. (Be able to compute set operations.) Using the three sets

$$\mathbb{X} = \{1, 2, 3, 4, 5\} \qquad \mathbb{Y} = \{0, 2, 4, 6, 8\} \qquad \mathbb{V} = \{0, 3, 5, 9, \}$$

over the universe of digits

$$\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

compute the following set operations

(a) $\mathbb{X} \cup \mathbb{Y}$
Answer: $\mathbb{X} \cup \mathbb{Y} = \{0, 1, 2, 3, 4, 5, 6, 8\}$.

(b) $\mathbb{X} \cup \overline{\mathbb{V}}$
Answer: $\mathbb{X} \cup \overline{\mathbb{V}} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(c) $(\mathbb{X} \cup \mathbb{Y}) \cap \mathbb{V}$
Answer: $(\mathbb{X} \cup \mathbb{Y}) \cap \mathbb{V} = \{0, 3, 4, 5\}$.

(d) $\overline{(\mathbb{X} \cup \mathbb{Y})} \cap \mathbb{V}$
Answer: $\overline{(\mathbb{X} \cup \mathbb{Y})} \cap \mathbb{V} = \{7\}$.

24. (Be able to compute set operations.) Using the three sets

$$\mathbb{X} = \{1, 2, 3, 4, 5\} \qquad \mathbb{Y} = \{0, 2, 4, 6, 8\} \qquad \mathbb{V} = \{0, 3, 5, 9, \}$$

over the universe of digits

$$\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

write expressions using set operators union $\cup$, intersection $\cap$ and set complement — that are equal to the following sets.

(a) $\mathbb{S} = \{6, 8\}$
Answer: $\mathbb{S} = (\mathbb{Y} - \mathbb{X}) - \mathbb{V} = \mathbb{Y} \cap \overline{\mathbb{X}} \cap \overline{\mathbb{V}}$

(b) $\mathbb{T} = \{1, 2, 4\}$
Answer: $\mathbb{T} = \mathbb{X} - \mathbb{V} = \mathbb{X} \cap \overline{\mathbb{V}}$

(c) $\mathbb{R} = \{0, 1, 2, 3, 4, 5, 6, 8\}$
Answer: $\mathbb{A} = \mathbb{X} \cup \mathbb{Y}$

(d) $\mathbb{W} = \{3, 5\}$
Answer: $\mathbb{W} = \mathbb{X} \cap \mathbb{V}$

25. (Be able to use the Cartesian product operator.) The Cartesian product $\mathbb{X} \times \mathbb{Y}$ is the set of ordered pairs

$$\mathbb{X} \times \mathbb{Y} = \{(x, y) : x \in \mathbb{X}, y \in \mathbb{Y}\}$$

(a) What is $\{a, b, c\} \times \{0, 1\}$?
Answer:

$$\{a, b, c\} \times \{0, 1\} = \{(a, 0), (a, 1), (b, 0), (b, 1), (c, 0), (c, 1)\}$$

(b) Is $\{a, b, c\} \times \{0, 1\} = \{0, 1\} \times \{a, b, c\}$?
Answer: No, the Cartesian product operator is not commutative.

(c) What is $|\{a, b, c\} \times \{0, 1\}|$?
Answer: $|\{a, b, c\} \times \{0, 1\}| = 6 = 3 \times 2$

(d) If $|\mathbb{X}| = m$ and $|\mathbb{Y}| = n$ what is $|\mathbb{X} \times \mathbb{Y}|$?
Answer: $|\mathbb{X} \times \mathbb{Y}| = mn$

(e) If $|\mathbb{X}| = m$ and $|\mathbb{Y}| = n$ how many subsets does $|\mathbb{X} \times \mathbb{Y}|$ have?
Answer: There are 2^{nm} subsets of $|\mathbb{X} \times \mathbb{Y}|$.

(f) Prove that $\mathbb{X} \times (\mathbb{Y} \cup \mathbb{V}) = (\mathbb{X} \times \mathbb{Y}) \cup (\mathbb{X} \times \mathbb{V})$.
Answer: Let $(x, b) \in \mathbb{X} \times (\mathbb{Y} \cup \mathbb{V})$. Then $x \in \mathbb{X}$ and $b \in \mathbb{Y} \cup \mathbb{V}$. That is, $x \in \mathbb{X}$ and $b \in \mathbb{Y}$ or $b \in \mathbb{V}$, so that $(x, b) \in (\mathbb{X} \times \mathbb{Y}) \cup (\mathbb{X} \times \mathbb{V})$. Conversely, let $(x, b) \in (\mathbb{X} \times \mathbb{Y}) \cup (\mathbb{X} \times \mathbb{V})$. then $(x, b) \in \mathbb{X} \times \mathbb{Y}$ or $(x, b) \in \mathbb{X} \times \mathbb{V}$. In either case, $x \in \mathbb{X}$ and $b \in \mathbb{Y} \cup \mathbb{V}$. Therefore, $(x, b) \in \mathbb{X} \times (\mathbb{Y} \cup \mathbb{V})$.

26. (Be able to use binomial coefficients to counting subsets.) How many k-element subsets of an n-element set are there? If there are 6 or fewer subsets list all of them.

(a) When $k = 1$ and $n = 2$
Answer: There are

$$\binom{2}{1} = \frac{2!}{1!1!} = 2$$

1-element subsets of a 2 element set. They are $\{0\}$, $\{1\}$.

(b) When $k = 3$ and $n = 4$
Answer: There are
$$\binom{4}{3} = \frac{4!}{3!1!} = 4$$
3-element subsets of a 4 element set. They are $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, and $\{1, 2, 3\}$.

(c) When $k = 2$ and $n = 5$
Answer: There are
$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$
2-element subsets of a 5 element set.

(d) When $k = 4$ and $n = 16$
Answer: There are
$$\binom{16}{4} = \frac{16!}{4!12!} = 1820$$
2-element subsets of a 5 element set.

(e) When $k = 51$ and $n = 52$
Answer: There are
$$\binom{52}{51} = \frac{52!}{51!1!} = 52$$
51-element subsets of a 52 element set.

(f) When $k = 5$ and $n = 13$
Answer: There are
$$\binom{13}{5} = \frac{13!}{5!8!} = 1286$$
5-element subsets of a 13 element set.

27. (Be able to use binomial coefficients to counting subsets.) The cardinality of $\mathbb{D}$ is 10, that is, $|\mathbb{D}| = 10$. Answer the following questions.

(a) How many subsets of $\mathbb{D}$ have no members?
Answer: There is one subset of $\mathbb{D}$ with no members: It is the empty set.

(b) How many subsets of $\mathbb{D}$ have 1 member?
Answer: There are 10 subset of $\mathbb{D}$ with 1 member.

(c) How many subsets of $\mathbb{D}$ have 2 member?
Answer: There are $\binom{10}{2} = (10 \cdot 9)/2 = 45$ subset of $\mathbb{D}$ with 2 members.

(d) How many subsets of $\mathbb{D}$ have 5 member?
Answer: There are $\binom{10}{5} = (10 \cdot 9 \cdot 8 \cdot 7 \cdot 5)/(5 \cdot 4 \cdot 3 \cdot 2) = 3 \cdot 2 \cdot 7 \cdot 5 = 210$ subset of $\mathbb{D}$ with 5 members.

(e) How many subsets of $\mathbb{D}$ have 7 member?
Answer: There are $\binom{10}{7} = (10 \cdot 9 \cdot 8)/(3 \cdot 2) = 5 \cdot 3 \cdot 8 = 120$ subset of $\mathbb{D}$ with 7 members.

(f) How many subsets of $\mathbb{D}$ have 10 members?
Answer: There is one subset of $\mathbb{D}$ with 10 members: It is the set $\mathbb{D}$ itself.

28. (Know Pascal's formula.) Fill in the missing binomial coefficient(s).

(a) $\binom{3}{2} = \_\_\_\_ + \binom{2}{1}$
Answer: $\binom{3}{2} = \binom{2}{2} + \binom{2}{1}$

(b) $\binom{7}{4} =$ ____ + ____
Answer: $\binom{7}{4} = \binom{6}{4} + \binom{6}{3}$

(c) ____ $= \binom{12}{8} + \binom{12}{7}$
Answer: $\binom{13}{8} = \binom{12}{8} + \binom{12}{7}$

(d) $\binom{n}{k} =$ ____ + ____
Answer: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

29. (Understand how Pascal's formula is derived.) The set $\mathbb{T}_3 = \{0, 1, 3\}$ has three subsets with 1 members:

$$\{0\}, \{1\}, \{3\}$$

and three subsets with 2 members:

$$\{0, 1\}, \{0, 3\}, \{1, 3\}$$

Show how to use these subsets to construct all 2 member subsets of

$$\mathbb{T}_4 = \{0, 1, 3, 6\} = \mathbb{T}_3 \cup \{6\}$$

and by doing so demonstrate that Pascal's formula

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

holds in this instance.

Answer: Each 1 element subset: $\{0\}$, $\{1\}$ and $\{3\}$ can be made into a 2 element subset of $\{0, 1, 3, 6\}$ by a union with the set $\{6\}$ yielding

$$\{0, 6\}, \{1, 6\}, \{3, 6\}$$

Each of the 2 element subsets: $\{0, 1\}$, $\{0, 3\}$ and $\{1, 3\}$ is a 2 element subset of $\mathbb{T}_4$. There are no other 2 element subsets of $\mathbb{T}_4$. Therefore

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

holds in this instance.

30. (Know the entries in a row of Pascal's triangle count the number of subsets of a given cardinality, and

that their sum is a power of 2.) Consider the rows of Pascal's triangle.

	Binomial Coefficients $\binom{n}{k}$									
	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(a) Row 0, $\langle 1 \rangle$, counts the number of subsets of the empty set $\emptyset$.

(b) Row 1, $\langle 1, 1 \rangle$, counts the number of subsets of the singleton set $\mathbb{Z}_1 = \{0\}$.

(c) Row 2, $\langle 1, 2, 1 \rangle$, counts the number of subsets of the binary set $\mathbb{Z}_2 = \{0, 1\}$.

i. How many subsets of $\mathbb{Z}_2$ have no, one, and two members?
Answer: There is $\binom{2}{0} = 1$ subset with no elements, there are $\binom{2}{1} = 2$ subset with one element, there is $\binom{2}{2} = 1$ subset with two element.

ii. How many subsets in total does $\mathbb{Z}_2$ have? Express your answer as a power of 2.
Answer: $\mathbb{Z}_2$ has $4 = 2^2$ subsets.

iii. What is $1 + 2 + 1$ equal to? Express your answer as a power of 2.
Answer: $1 + 2 + 1 = 4 = 2^2$.

(d) Row 3, $\langle 1, 3, 3, 1 \rangle$, counts the number of subsets of the ternary set $\mathbb{Z}_3 = \{0, 1, 2\}$.

i. How many subsets of $\mathbb{Z}_3$ have no, one, two, and three members?
Answer: There is $\binom{3}{0} = 1$ subset with no elements, there are $\binom{3}{1} = 3$ subset with one element, there are $\binom{3}{2} = 3$ subset with two element, there are $\binom{3}{1} = 1$ subset with three element.

ii. How many subsets in total does $\mathbb{Z}_3$ have? Express your answer as a power of 2.
Answer: $\mathbb{Z}_3$ has $8 = 2^3$ subsets.

iii. What is $1 + 3 + 3 + 1$ equal to? Express your answer as a power of 2.
Answer: $1 + 3 + 3 + 1 = 8 = 2^3$.

iv. How many subsets of $\mathbb{Z}_n$ have no, one, two, $n - 2$, $n - 1$ and n members?
Answer: There is $\binom{n}{0} = 1$ subset with no elements, there are $\binom{n}{1} = n$ subset with one element, there are $\binom{n}{2} = n(n-1)/2$ subset with two element, there are $\binom{n}{2} = n(n-1)/2$ subset with two element, there are $\binom{n}{n-2} = n(n-1)/2$ subset with $n - 2$ element. there are $\binom{n}{n-1} = n$ subset with $n - 1$ element. there is $\binom{n}{n} = 1$ subset with n elements.

31. (Know the relationship between columns in Pascal's rectangle.) Consider the columns of Pascal's rectangle: Fill the upper triangle of Pascal's triangle with 0's.

	Binomial Coefficients $\binom{n}{k}$									
	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0	0
5	1	5	10	10	5	1	0	0	0	0
6	1	6	15	20	15	6	1	0	0	0
7	1	7	21	35	35	21	7	1	0	0
8	1	8	28	56	70	56	28	8	1	0
9	1	9	36	84	126	126	84	36	9	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(a) Column 0 is the Alice sequence: $\vec{A} = \langle a_0, a_1, a_2, a_3 \ldots \rangle = \langle 1, 1, 1, 1, 1, 1, \ldots \rangle$.

i. The Alice sequence has an initial value a_0. What is it?
Answer: The initial value is $a_0 = 1$.

ii. What function $a(n)$ computes terms in the Alice sequence?
Answer: The constant function $a(n) = 1$ computes the terms in the Alice sequence.

iii. The Alice sequence has recurrence equation that expresses a_n in terms of a_{n-1}. What is the recurrence equation?
Answer: In subscript notation, the recurrence equation is

$$a_n = a_{n-1}$$

In function notation it is

$$a(n) = a(n-1)$$

(b) Column 1 is the Gauss sequence: $\vec{G} = \langle g_0, g_1, g_2, g_3 \ldots \rangle = \langle 0, 1, 2, 3, 4, 5, 6, \ldots \rangle$.

i. The Gauss sequence has an initial value g_0. What is it?
Answer: The initial value is $g_0 = 0$.

ii. What function $g(n)$ computes terms in the Gauss sequence?
Answer: The linear function $g(n) = n$ computes the terms in the Gauss sequence.

iii. The Gauss sequence has recurrence equation that expresses g_n in terms of g_{n-1}. What is the recurrence equation?
Answer: In subscript notation, the recurrence equation is

$$g_n = g_{n-1} + 1 = g_{n-1} + a_{n-1}$$

In function notation it is

$$g(n) = g(n-1) + a(n-1)$$

(c) Column 2 is the Triangular sequence: $\vec{T} = \langle t_0, t_1, t_2, t_3 \ldots \rangle = \langle 0, 0, 1, 3, 6, 10, 15, \ldots \rangle$.

i. The Triangular sequence has an initial value t_0. What is it?
Answer: The initial value is $t_0 = 0$.

ii. What function $t(n)$ computes terms in the Triangular sequence?
Answer: The quadratic function $t(n) = n(n-1)/2$ computes the terms in the Triangular sequence.

iii. The Triangular sequence has recurrence equation that expresses t_n in terms of t_{n-1}. What is the recurrence equation?
Answer: In subscript notation, the recurrence equation is

$$t_n = t_{n-1} + (n-1) = t_{n-1} + g_{n-1}$$

In function notation it is

$$t(n) = t(n-1) = t(n-1) + g(n-1)$$

32. (Be able to count the number of operations in a computation.) What is the "time" taken to compute the binomial coefficient using the factorial formula?

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Assume that "time" is measured by the number of multiplications in the reduced formula

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2}$$

Answer: Computing the numerator requires $k-1$ multiples. Computing the denominator requires $k-2$ multiples. Therefore, the "time" complexity is $2k-3$ or $2k-2$ if you count the divide.

33. (Be able to count the number of operations in a computation.) What is the "time" taken to compute the binomial coefficient using the recursive (Pascal's) formula?

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}?$$

Assume that "time" is measured by the minimal number of additions required to build Pascal's triangle to $\binom{n}{k}$. In particular, you may assume that $k = \min\{k, n-k\}$ and use the symmetry of the triangle.

Answer: It is useful build Pascal's triangle and identify the values that must be computed.

	Binomial Coefficients $\binom{n}{k}$									
	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

Therefore, the "time" complexity is the sum of natural numbers from 2 to k, or $k(k+1)/2$.

34. (Know that reasoning about sets is complex.) Consider the "set" $\mathbb{S} = \{x : x \notin x\}$.

(a) Show that $\mathbb{S} \in \mathbb{S}$ is a contradiction.
Answer: If $\mathbb{S} \in \mathbb{S}$, that is, if $\mathbb{S}$ is an element of $\mathbb{S}$, then by the definition of $\mathbb{S}$, $\mathbb{S} \notin \mathbb{S}$. A contradiction.

(b) Show that $S \notin \mathbb{S}$ is a contradiction.
Answer: If $\mathbb{S} \notin \mathbb{S}$, that is, if $\mathbb{S}$ is not an element of $\mathbb{S}$, then by the definition of $\mathbb{S}$, $\mathbb{S} \in \mathbb{S}$. A contradiction.

This is known as Russell's paradox.

References

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