CSE 1400 Applied Discrete Mathematics
Relations and Functions

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Spring 2012

Problems labeled

⚠ are important and must be mastered.

are harder and cover interesting advanced material that is not properly part of this course.

○ are tangential to the course but cover material that is useful elsewhere.

Problems on Relations

Background

A (binary) relation is a set of ordered pairs

$$\{(x, y) : x \sim y\}$$

where $\sim$ is a relational symbol, for example, equal ($=$), less than ($<$), divides ($|$), congruence mod n ($\equiv \pmod{n}$), subset ($\subseteq$), et cetera.

Two important types of relations are equivalences and (partial) orders:

- Equivalence relations are
  - reflexive $(\forall x)(x \sim x)$,
  - symmetric $(\forall x, y)(x \sim y \to y \sim x)$, and
  - transitive $(\forall x, y, z)((x \sim y) \wedge (y \sim z) \to x \sim z)$.
- Partial orders are reflexive, transitive, and antisymmetric

  $$(\forall x, y)((x \sim y) \wedge (y \sim x) \to x = y)$$

  or by contraposition

  $$(\forall x, y)(x \neq y \to (x \not\sim y) \vee (y \not\sim x))$$

1. ⚠ (Recognize whether several important relations are reflexive or not.) Which of the following statements are `True` and which are `False`?

   (a) $(\forall x \in \mathbb{R})(x = x)$.
   Answer: This is `True`. Equality is a reflexive relation.

   (b) $(\forall x \in \mathbb{R})(x \neq x)$.
   Answer: This is `False`. Not equal is not a reflexive relation.

   (c) $(\forall x \in \mathbb{R})(x < x)$.
   Answer: This is `False`. Less than is not a reflexive relation.

   (d) $(\forall x \in \mathbb{R})(x \geq x)$.
   Answer: This is `True`. Greater than or equal is a reflexive relation.

   (e) $(\forall a \in \mathbb{N})(a \mid a)$.
   Answer: This is `True`. Divides is a reflexive relation.

   (f) $(\forall \mathbb{X} \in 2^{\mathbb{U}})(\mathbb{X} \subseteq \mathbb{X})$.
   Answer: This is `True`. Subset is a reflexive relation.

(g) Let $n \in \mathbb{N}$, $n > 1$ be fixed. $(\forall a \in \mathbb{Z})(a \equiv a \pmod{n})$.
Answer: This is `True`. Congruence mod n is a reflexive relation.

(h) $(\forall a \in \mathbb{Z})(\gcd(a, a) = 1)$
Answer: This is `False`. The greatest common divisor of a and a is $|a|$, which is most often not equal to 1. That is, relatively prime is not a reflexive relation.

2. (Recognize whether several important relations are symmetric or not.) Which of the following statements are `True` and which are `False`?

(a) $(\forall x, y \in \mathbb{R})((x = y) \to (y = x))$.
Answer: This is `True`. Equality is symmetric.

(b) $(\forall x, y \in \mathbb{R})((x \neq y) \to (y \neq x))$.
Answer: This is `True`. Not equal is symmetric

(c) $(\forall x, y \in \mathbb{R})(x < y \to y < x)$.
Answer: This is `False`. Less than is not symmetric.

(d) $(\forall x, y \in \mathbb{R})(x \geq y \to y \geq x)$.
Answer: This is `False`. Greater than or equal is not symmetric.

(e) $(\forall a, b \in \mathbb{N})(a \mid b \to b \mid a)$.
Answer: This is `False`. Divides is not symmetric.

(f) $(\forall \mathbb{X}, \mathbb{Y} \in 2^{\mathbb{U}})(\mathbb{X} \subseteq \mathbb{Y} \to \mathbb{Y} \subseteq \mathbb{X})$.
Answer: This is `False`. Subset is not symmetric.

(g) Let $n \in \mathbb{N}$, $n > 1$ be fixed. $(\forall a, b \in \mathbb{Z})(a \equiv b \pmod{n} \to b \equiv a \pmod{n})$.
Answer: This is `True`. Congruence mod n is symmetric.

(h) $(\forall a, b \in \mathbb{Z})(\gcd(a, b) = 1 \to \gcd(b, a) = 1)$.
Answer: This is `True`. Relatively prime is symmetric.

3. (Recognize whether several important relations are antisymmetric or not.) Which of the following statements are `True` and which are `False`?

(a) $(\forall x, y \in \mathbb{R})(((x = y) \wedge (y = x)) \to (x = y))$.
Answer: This is `True`. Equality is antisymmetric.

(b) $(\forall x, y \in \mathbb{R})((x \neq y \wedge y \neq x) \to (x = y))$.
Answer: This is `False`. Not equal in not antisymmetric

(c) $(\forall x, y \in \mathbb{R})((x < y \wedge y < x) \to (x = y))$.
Answer: This is `True`. It is `True` by virtue of its logical form: $\mathrm{P} \to \mathrm{Q}$ where the premise $\mathrm{P} = (x < y \wedge y < x)$ is always `False` which makes the conditional always `True`.

(d) $(\forall x, y \in \mathbb{R})(x \geq y \wedge y \geq x \to x = y)$.
Answer: This is `True`. Greater than or equal is antisymmetric.

(e) $(\forall a, b \in \mathbb{N})(a \mid b \wedge b \mid a \to a = b)$.
Answer: This is `True`. Divides is antisymmetric.

(f) $(\forall \mathbb{X}, \mathbb{Y} \in 2^{\mathbb{U}})((\mathbb{X} \subseteq \mathbb{Y}) \wedge (\mathbb{Y} \subseteq \mathbb{X}) \to \mathbb{X} = \mathbb{Y})$.
Answer: This is `True`. Subset is antisymmetric.

(g) Let $n \in \mathbb{N}$, $n > 1$ be fixed. $(\forall a, b \in \mathbb{Z})(a \equiv b \pmod{n} \wedge b \equiv a \pmod{n} \to a = b)$.
Answer: This is `False`. Congruence mod n is not antisymmetric.

(h) $(\forall a, b \in \mathbb{Z})((\gcd(a, b) = 1) \wedge (\gcd(b, a) = 1) \rightarrow a = b)$.
Answer: This is `False`. Relatively prime is not antisymmetric.

4. ⚠ (Recognize whether several important relations are transitive or not.) Which of the following statements are `True` and which are `False`?

(a) $(\forall x, y, z \in \mathbb{R})(((x = y) \wedge (y = z)) \rightarrow (x = z))$.
Answer: This is `True`. Equality is transitive.

(b) $(\forall x, y, z \in \mathbb{R})((x \neq y \wedge y \neq z) \rightarrow (x \neq z))$.
Answer: This is `False`. Not equal is not transitive. A counterexample to the statement that not equal is transitive is: $x = 0$, $y = 1$ and $z = 0$.

(c) $(\forall x, y, z \in \mathbb{R})((x < y \wedge y < z) \rightarrow (x < z))$.
Answer: This is `True`. Less than is transitive.

(d) $(\forall x, y, z \in \mathbb{R})(x \geq y \wedge y \geq z \rightarrow x \geq z)$.
Answer: This is `True`. Greater than or equal is transitive.

(e) $(\forall a, b, c \in \mathbb{N})(a \mid b \wedge b \mid c \rightarrow a \mid c)$.
Answer: This is `True`. If $a \mid b$, then there is a natural number d such that $ad = b$. And, If $b \mid c$, then there is a natural number e such that $be = c$. Therefore $a(de) = (ad)e = be = c$ and a divides c. Divides is transitive.

(f) $(\forall \mathbb{X}, \mathbb{Y}, \mathbb{Z} \in 2^{\mathbb{U}})((\mathbb{X} \subseteq \mathbb{Y}) \wedge (\mathbb{Y} \subseteq \mathbb{Z}) \rightarrow \mathbb{X} \subseteq \mathbb{Z})$.
Answer: This is `True`. Subset is transitive.

(g) Let $n \in \mathbb{N}$, $n > 1$ be fixed. $(\forall a, b, c \in \mathbb{Z})(a \equiv b \pmod{n} \wedge b \equiv c \pmod{n} \rightarrow a \equiv c \pmod{n})$.
Answer: This is `True`. Congruence mod n is transitive. If $a \equiv b \pmod{n}$, then $a - b$ is a multiple of n. And, if $b \equiv c \pmod{n}$, then $b - c$ is a multiple of n. Therefore $(a - b) + (b - c) = a - c$ is a multiple of n.

(h) $(\forall a, b, c \in \mathbb{Z})((\gcd(a, b) = 1) \wedge (\gcd(b, c) = 1) \rightarrow \gcd(a, c) = 1)$.
Answer: This is `False`. Relatively prime is not transitive. A counterexample is $a = 2$, $b = 3$ and $c = 4$.

5. (Be able to describe the relations determined by Boolean operations.) Describe these relations over the set of Boolean variables.

(a) $\mathrm{P} \wedge \mathrm{Q} =$ `True`
Answer: This is the set $\{(1, 1)\}$.

(b) $\mathrm{P} \wedge \mathrm{Q} =$ `False`
Answer: This is the set $\{(0, 0), (0, 1), (1, 0)\}$.

(c) $\mathrm{P} \vee \mathrm{Q} =$ `True`
Answer: This is the set $\{(0, 1), (1, 0), (1, 1)\}$.

(d) $\mathrm{P} \vee \mathrm{Q} =$ `False`
Answer: This is the set $\{(0, 0)\}$.

(e) $\mathrm{P} \rightarrow \mathrm{Q} =$ `True`
Answer: This is the set $\{(0, 0), (0, 1), (1, 1)\}$.

(f) $\mathrm{P} \rightarrow \mathrm{Q} =$ `False`
Answer: This is the set $\{(1, 0)\}$.

(g) $\mathrm{P} \equiv \mathrm{Q} =$ `True`
Answer: This is the set $\{(0, 0), (1, 1)\}$.

(h) $\text{P} \equiv \text{Q} =$ `False`
Answer: This is the set $\{(0, 1), (1, 0)\}$.

(i) $\text{P} \oplus \text{Q} =$ `True`
Answer: This is the set $\{(0, 1), (1, 0)\}$.

(j) $\text{P} \oplus \text{Q} =$ `False`
Answer: This is the set $\{(0, 0), (1, 1)\}$.

6. (Be able to construct all relations in small cases.) List relations in the following instances.

(a) The relations from $\{a\}$ to the set of bits $\mathbb{B}$.
Answer:

i. $\emptyset$
ii. $\{(a, 0)\}$
iii. $\{(a, 1)\}$
iv. $\{(a, 0), (a, 1)\}$

(b) List all relations on the set of bits $\mathbb{B}$.
Answer:

i. $\emptyset$
ii. $\{(0,0)\}$
iii. $\{(0,1)\}$
iv. $\{(1,0)\}$
v. $\{(1,1)\}$
vi. $\{(0,0),(0,1)\}$
vii. $\{(0,0),(1,0)\}$
viii. $\{(0,0),(1,1)\}$
ix. $\{(0,1),(1,0)\}$
x. $\{(0,1),(1,1)\}$
xi. $\{(1,0),(1,1)\}$
xii. $\{(0,0),(0,1),(1,0)\}$
xiii. $\{(0,0),(0,1),(1,1)\}$
xiv. $\{(0,0),(1,0),(1,1)\}$
xv. $\{(0,1),(1,0),(1,1)\}$
xvi. $\{(0,0),(0,1),(1,0),(1,1)\}$

7. (Be able to recognize properties of relations from there representation as ordered pairs.) Which of the relations in problem 6b are reflexive, which are symmetric, which are antisymmetric, and which are transitive?
Answer:

- Reflexive: 8, 13, 14, 16
- Symmetric: 1, 2, 5, 8, 9, 12, 15, 16
- Antisymmetric: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14
- Transitive: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16

8. ⚠ (Be able to represent a relation as an adjacency matrix.) A relation on an n-element set $\mathbb{A} = \{a_0, a_1, \ldots, a_{n-1}\}$ can be represented by an $n \times n$ adjacency matrix of Boolean values.

$$\left[\begin{array}{c|cccc} & a_0 & a_1 & \cdots & a_{n-1} \\ \hline a_0 & b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} \\ a_1 & b_{1,0} & b_{1,1} & \cdots & b_{1,n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-1} \end{array}\right]$$

Where the entries $b_{i,j} \in \mathbb{B}$ are Boolean values such that $b_{i,j} = 1$ if a_i is related to a_j and $b_{i,j} = 0$ otherwise. Construct the adjacency matrix for the following relations.

(a) Equality on the octal numerals.
Answer:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0
3	0	0	0	1	0	0	0	0
4	0	0	0	0	1	0	0	0
5	0	0	0	0	0	1	0	0
6	0	0	0	0	0	0	1	0
7	0	0	0	0	0	0	0	1

(b) Divides on the octal numerals.
Answer:

	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	0	0	1	0	1	0	1	0
3	0	0	0	1	0	0	1	0
4	0	0	0	0	1	0	0	0
5	0	0	0	0	0	1	0	0
6	0	0	0	0	0	0	1	0
7	0	0	0	0	0	0	0	1

(c) Congruence mod 4 on the octal numerals.
Answer:

	0	1	2	3	4	5	6	7
0	1	0	0	0	1	0	0	0
1	0	1	0	0	0	1	0	0
2	0	0	1	0	0	0	1	0
3	0	0	0	1	0	0	0	1
4	1	0	0	0	1	0	0	0
5	0	1	0	0	0	1	0	0
6	0	0	1	0	0	0	1	0
7	0	0	0	1	0	0	0	1

9. (Know some geometric relations.) Consider lines written in slope-intercept form $y = mx + b$ and standard form $ax + by + c = 0$.

(a) Describe what must be true for two lines, written as slope-intercept pairs (m_0, b_0) and (m_1, b_1), to be perpendicular. Use the notation

$$(m_0, b_0) \perp (m_1, b_1)$$

to denote the perpendicular relationship between two lines.
Answer: Two lines are perpendicular when their slopes are "negative reciprocals," that is $m_0 \cdot m_1 = -1$. Therefore, $(m_0, b_0) \perp (m_1, b_1)$ when $m_0 = -1/m_1$. There is a special case, when the slope of one line is zero and the other line has undefined, often called infinite, slope. This case could be handled using calculus and the idea of LIMITS.

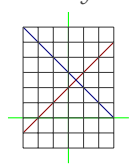
(b) Describe what must be true for two lines, written as slope-intercept pairs (m_0, b_0) and (m_1, b_1), to be parallel. Use the notation

$$(m_0, b_0) \| (m_1, b_1)$$

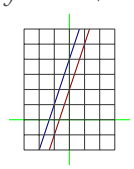
to denote the parallel relationship between two lines.
Answer: Two lines are parallel when they have equal slopes. Therefore, $(m_0, b_0) \| (m_1, b_1)$ when $m_0 = m_1$.

(c) Interpret the following ordered pairs as slope-intercept descriptions of lines. Which pairs are perpendicular? Which pairs are parallel?

i. $(1, 2)$ and $(-1, 3)$.
Answer: These pairs describe perpendicular lines: $y = x + 2$ and $y = -x + 3$.



ii. $(3, 2)$ and $(3, 4)$.
Answer: These pairs describe parallel lines: $y = 3x + 2$ and $y = 3x + 4$.



iii. $(-0.5, 2)$ and $(2, 3)$.
Answer: These pairs describe perpendicular lines: $y = 0.5x + 2$ and $y = 2x + 3$.

iv. $(-3, 2)$ and $(4, 3)$.
Answer: These pairs describe lines that are neither perpendicular nor parallel: $y = -3x + 2$ and $y = 4x + 3$.

(d) Describe what must be true for two lines, written in standard form (a_0, b_0, c_0) and (a_1, b_1, c_1), to be perpendicular. Use the notation

$$(a_0, b_0, c_0) \perp (a_1, b_1, c_1)$$

to denote the perpendicular relationship between two lines.
Answer: The two lines are perpendicular if the inner product $\vec{a_0, b_0} \cdot \vec{a_1, b_1} = a_0 a_1 + b_0 b_1$ is equal to zero.

(e) The set of all points (x, y) that lie on a line through the origin can be written in the form

$$ax + by = 0 \quad \text{where } (a \neq 0) \vee (b \neq 0)$$

i. What is a logically equivalent way to write the clause $(a \neq 0) \vee (b \neq 0)$?
Answer: $\neg((a = 0) \wedge (b = 0))$.

ii. Consider $\langle a, b\rangle$ to be the tip of a vector from the origin. What is the Euclidean length of $\langle a, b\rangle$.
Answer: The length is $|\langle a, b\rangle| = \sqrt{a^2 + b^2}$.

iii. Say two points (x_1, y_1) and (x_2, y_2) are equivalent if they lie on the same line through the origin.

A. Is this relation reflexive?
Answer: Yes.

B. Is this relation symmetric?
Answer: Yes.

C. Is this relation antisymmetric?
Answer: No.

D. Is this relation transitive?
Answer: Yes.

10. (Be able to understand the approximately equal relation.) Say that two real numbers x and y are approximately equal if the differ by $\epsilon = 1/32 = 3.125 \times 10^{-2}$.

$$x \approx y \quad \text{if and only if } |x - y| \leq \epsilon$$

(a) Is approximately equal reflexive? Explain.
Answer: Yes. $x \approx x$ for all x since $|x - x| = 0 \leq \epsilon$

(b) Is approximately equal symmetric? Explain.
Answer: Yes. If $x \approx y$, then $y \approx x$ since $|x - y| \leq \epsilon \implies |y - x| \leq \epsilon$

(c) Is approximately equal antisymmetric? Explain.
Answer: No, $x \approx y$ and $y \approx x$ does not imply $x = y$, As a counterexample $\pi \approx 3.14$ and $3.14 \approx \pi$ but $\pi \neq 3.14$

(d) Is approximately equal transitive? Explain.
Answer: No, if $x \approx y$ and $y \approx z$, then x need not be approximately equal to z. As a counterexample $3.14 \approx \pi$ and $\pi \approx 3.17275$ but $3.14 \not\approx 3.17275$. since $|3.17275 - 3.14| > 3.125 \times 10^{-2}$

11. (Be able to interpret and represent a simple relation.) In the childhood game Rock, Paper, Scissors "rock beats scissors, scissors beat paper, and paper beats rock."

(a) Construct the adjacency matrix that represents the "beats" relation.
Answer:

	Rock	Paper	Scissors
Rock	0	0	1
Paper	1	0	0
Scissor	0	1	0

(b) Is "beats" reflexive, symmetric, antisymmetric, transitive?
Answer: "beats" is antisymmetric and that is all.

12. (Understand the predicate logic form of reflexive.) Negate the definition of "relation $\sim$ is reflexive" to define "relation $\sim$ is not reflexive"

Answer: A relation is reflexive when

$$(\forall x \in \mathbb{X})(x \sim x)$$

Negating this statement gives: A relation is not reflexive when

$$\neg(\forall x \in \mathbb{X})(x \sim x) = (\exists x \in \mathbb{X})(x \not\sim x)$$

13. (Understand the predicate logic form of symmetric.) Negate the definition of "relation $\sim$ is symmetric:" to define "relation $\sim$ is not symmetric."
Answer: A relation is symmetric when

$$(\forall x, y \in \mathbb{X})((x \sim y) \implies (y \sim x))$$

Negating this statement gives: A relation is not symmetric, called asymmetric, when

$$(\exists x, y \in \mathbb{X})((x \sim y) \wedge (y \not\sim x))$$

14. (Understand the predicate logic form of antisymmetric.) What is the contraposition of the antisymmetry statement

$$(\forall x, y \in \mathbb{X})(((x \sim y) \wedge (y \sim x)) \to (x = y))?$$

Answer:

$$(\forall x, y \in \mathbb{X})((x \neq y) \to ((x \not\sim y) \vee (y \not\sim x)))$$

15. (Understand the predicate logic form of antisymmetric.) Negate the definition of "relation $\sim$ is antisymmetric:" to define "relation $\sim$ is not antisymmetric."
Answer: A relation is antisymmetric when

$$(\forall x, y \in \mathbb{X})(((x \sim y) \wedge (y \sim x)) \to (x = y))$$

Negating this statement gives: A relation is not antisymmetric when

$$(\exists x, y \in \mathbb{X})(((x \sim y) \vee (y \sim x)) \wedge (x \neq y))$$

Equivalence Relations

Background

Partitioning a set into subsets of elements that have a common property creates an equivalence relation.

1. (Know the what a partition of a set is.) Answer `True` or `False`: A partition of a set $\mathbb{X}$ is a collection of non-empty, pairwise disjoint subsets of $\mathbb{X}$ whose union is $\mathbb{X}$.
Answer: This is `True`.
2. (Be able to construct all partitions of small sets.)
   (a) What are the partitions of $\mathbb{B}$?
   Answer: There is one partition into one subset: $\{\{0, 1\}\}$. There is one partition into two subsets: $\{\{0\}, \{1\}\}$.

(b) What are the partitions of $\{0, 1, 2\}$?
Answer: There is one partition into one subset: $\{\{0, 1, 2\}\}$. There are three partition into two subsets: $\{\{0, 1\}, \{2\}\}$, $\{\{0, 2\}, \{1\}\}$, $\{\{1, 2\}, \{0\}\}$. There is one partition into one subset: $\{\{0\}, \{1\}, \{2\}\}$.

(c) What are the partitions of $\{0, 1, 2, 3\}$?
Answer:

i. There is one partition into one subset: $\{\{0, 1, 2, 3\}\}$.

ii. There are seven partition into two subsets: $\{\{0, 1, 2\}, Set3\}$. $\{\{0, 1\}, \{2, 3\}\}$, $\{\{0, 2\}, \{1, 3\}\}$, $\{\{1, 2\}, \{0, 3\}\}$, $\{\{0, 1, 3\}, \{2\}\}$, $\{\{0, 2, 3\}, \{1\}\}$, $\{\{1, 2, 3\}, \{0\}\}$.

iii. There is six partition into three subsets: $\{\{0, 1\}, \{2\}, \{3\}\}$, $\{\{0, 2\}, \{1\}, \{3\}\}$, $\{\{1, 2\}, \{0\}, \{3\}\}$, $\{\{0, 3\}, \{1\}, \{2\}\}$, $\{\{0\}, \{1, 3\}, \{2\}\}$, $\{\{0\}, \{1\}, \{2, 3\}\}$.

iv. There is one partition into four subset: $\{\{0, \}, \{1\}, \{2\}, \{4\}\}$.

3. ⚠ (Know that Stirling number of the second kind $\left\{ {n \atop m} \right\}$ counts the number partitions of an n element set into m subsets.) Using the results from problem 2 compute the following Stirling number of the second kind.

(a) $\left\{ {2 \atop 1} \right\}$
Answer: There is $\left\{ {2 \atop 1} \right\} = 1$ partition of a 2 element set into 1 subset.

(b) $\left\{ {2 \atop 2} \right\}$
Answer: There is $\left\{ {2 \atop 2} \right\} = 1$ partition of a 2 element set into 2 subsets.

(c) $\left\{ {3 \atop 1} \right\}$
Answer: There is $\left\{ {3 \atop 1} \right\} = 1$ partition of a 3 element set into 1 subset.

(d) $\left\{ {3 \atop 2} \right\}$
Answer: There are $\left\{ {3 \atop 2} \right\} = 3$ partitions of a 3 element set into 2 subsets.

(e) $\left\{ {3 \atop 3} \right\}$
Answer: There is $\left\{ {3 \atop 3} \right\} = 1$ partition of a 3 element set into 3 subsets.

(f) $\left\{ {4 \atop 1} \right\}$
Answer: There is $\left\{ {4 \atop 1} \right\} = 1$ partition of a 4 element set into 1 subset.

(g) $\left\{ {4 \atop 2} \right\}$
Answer: There are $\left\{ {4 \atop 2} \right\} = 7$ partitions of a 4 element set into 2 subsets.

(h) $\left\{ {4 \atop 3} \right\}$
Answer: There are $\left\{ {4 \atop 3} \right\} = 6$ partitions of a 4 element set into 3 subsets.

(i) $\left\{ {4 \atop 4} \right\}$
Answer: There is $\left\{ {4 \atop 4} \right\} = 1$ partition of a 4 element set into 4 subsets.

4. (Know the recursion for Stirling numbers of the second kind.) Stirling's triangle of the second kind is

Stirling Numbers of the Second Kind $\left\{ {n \atop m} \right\}$

		Subset m								
		0	1	2	3	4	5	6	7	8
	0	1	0	0	0	0	0	0	0	0
	1	0	1	0	0	0	0	0	0	0
	2	0	1	1	0	0	0	0	0	0
	3	0	1	3	1	0	0	0	0	0
n	4	0	1	7	6	1	0	0	0	0
	5	0	1	15	25	10	1	0	0	0
	6	0	1	31	90	65	15	1	0	0
	7	0	1	63	301	350	140	21	1	0
	8	0	1	127	966	1701	1050	266	28	1

What is the recurrence equation and boundary conditions for the entry $\left\{ {n \atop m} \right\}$ in row n, column m?
Answer: The entry $\left\{ {n \atop m} \right\}$ can be computed by the recurrence equation

$$\left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}$$

with boundary conditions $(\forall n \in \mathbb{N})(\left\{ {n \atop n} \right\} = 1)$, $(\forall n \in \mathbb{N},\, n > 0)(\left\{ {n \atop 0} \right\} = 1)$.

5. (Know that equality is an equivalence relation.) Is equality an equivalence relation on the set of real numbers?
Answer: Yes. Equality is reflexive $(\forall x \in \mathbb{R})(x = x)$, symmetric $(\forall x, y \in \mathbb{R})((x = y) \to (y = x)$, and transitive $(\forall x, y, z \in \mathbb{R})((x = y) \wedge (y = z) \to (x = z)$.

6. (Know that congruence mod n is an equivalence relation.) Let a and b be in integers ($a, b \in \mathbb{Z}$), and let n be a natural number. Say a is congruent to b mod n if $a - b = cn$ for some integer $c \in \mathbb{Z}$. Prove that congruence mod n an equivalence relation?
Answer: Yes. Congruence mod n is reflexive $(\forall x \in \mathbb{Z})(x - x = 0 \cdot n)$, symmetric $(\forall x, y \in \mathbb{R})((x - y = cn) \to (y - x = -cn))$, and transitive $(\forall x, y, z \in \mathbb{R})((x - y = cn) \wedge (y - z = dn) \to (x - z = (c + d)n))$.

7. (Know that equal magnitude is an equivalence relation.) Let $a, b \in \mathbb{Z}$ and say a and b have equal magnitude if their absolute values are equal, that is $|a| = |b|$. Prove that "equal magnitude" is an equivalence relation.
Answer: Equal magnitude is reflexive: $|a| = |a|$. Equal magnitude is symmetric: if $|a| = |b|$, then $|b| = |a|$. Equal magnitude is transitive: if $|a| = |b|$ and $|b| = |c|$, then $|a| = |c|$. Equal magnitude partitions the non-zero integers into pairs:

$$\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\}, \ldots \{n, -n\}, \ldots$$

8. (Be able to show a relation is an equivalence.) On the set of real numbers $\mathbb{R} = \{x : -\infty < x < \infty\}$ define the relation

$$\mathbb{S} = \{(x, y) : x, y \in \mathbb{R}, \text{ and } x - y \text{ is an integer}\}$$

Show that $\mathbb{S}$ is an equivalence relation on $\mathbb{R}$.

Answer: There are three things to show.

(a) The relation is reflexive: For every real number x, $x - x$ is the integer 0.

(b) The relation is symmetric: For all real numbers x and y, if $x - y$ is an integer, then $y - x$ is an integer.

(c) The relation is transitive: For all real numbers x, y, and z, if $x - y$ is an integer and $y - z$ is an integer, then $(x - y) + (y - z) = x - z$ is an integer.

This equivalence has infinitely many equivalence classes. For each $x \in \mathbb{R}$, the equivalence class for x is

$$[x] = \{x + k : k \in \mathbb{Z}\}$$

9. ⚠ (Know that equality of rational values is an equivalence regardless of the representation of the rational.) Let (a, b) and (c, d) be ordered pairs of integers, excluding the origin, that is $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$. Show that

$$(a, b) \sim (c, d) \text{ if } ad = bc$$

is an equivalence relation. (Think of $r = \frac{a}{b}$ and $s = \frac{c}{d}$ as rational numbers.) If the cross-product ad and cb are equal say r is equivalent to s and write $r \equiv s$.

(a) Give instances of $r = (a, b)$ and $s = (c, d)$ that are equivalent.
Answer: The pairs $r = (3, 4)$ and $s = (12, 16)$ are equivalent.

(b) Give instances of $r = (a, b)$ and $s = (c, d)$ that are not equivalent.
Answer: The pairs $r = (3, 4)$ and $s = (-6, -15)$ are not equivalent.

(c) What pairs are equivalent to the pair $(1, 1)$?
Answer: Every-and-only pairs of the form (a, a) are equivalent to $(1, 1)$.

(d) Is is $r \equiv s$ reflexive? Explain.
Answer: The relation is reflexive since $r \equiv r$ for all ordered pairs $r = (a, b)$. That is, the cross-products ab and ab are equal.

(e) Is $r \equiv s$ symmetric? Explain.
Answer: The relation is symmetric since if $r \equiv s$ then $s \equiv r$ for all ordered pairs $r = (a, b)$ and $s = (c, d)$. That is, if $ad = cb$, then $cb = ad$.

(f) Is $r \equiv s$ transitive? Explain.
Answer: The relation is transitive since if $r \equiv s$ and $s \equiv u$ for all ordered pairs $r = (a, b)$, $s = (c, d)$, and $u = (e, f)$, then $ad = cb$ and $cf = de$. Therefore, $adf = bcf = bde$ and $af = be$ provided $d \neq 0$. If $d = 0$, then $b = 0$ and $f = 0$, so the equation $af = be$ is also satisfied.

(g) Is $r \equiv s$ an equivalence relation? Explain.
Answer: Yes. A relation that is reflexive, symmetric, and transitive is, by definition, an equivalence relation.

(h) How does this relation describe equivalent names for rational fractions?
Answer: For $b \neq 0$, interpret $r = (a, b)$ as the rational a/b. Two pairs r and $s = (c, d)$ are equivalent when the rationals a/b and c/d are equal. For example, $(3, 4) \equiv (6, 8)$ because $3/4 = 6/8$. When $b = 0$, interpret $r = (a, 0)$ as some thing that we'll not define, but simply call infinity and denote by ∞.

(i) How does this relation describe a geometric relationship between points in a two-dimensional space?

10. (Be able to show an equivalence between ordered pairs of values.) Let a, b, c, d be integers and say the ordered pair (a, b) related to (c, d) if $a + d = b + c$.

$$(a, b) \sim (c, d) \quad \text{if } a + d = b + c$$

(a) Is $\sim$ reflexive? Explain.
Answer: Yes. $(a, b) \sim (a, b)$ for all integers a and b since $a + b = b + a$

(b) Is $\sim$ symmetric? Explain.
Answer: Yes. If $(a, b) \sim (c, d)$, then $a + d = b + c$ $(c, d) \sim (a, b)$, then $c + b = d + a$

(c) Is $\sim$ transitive? Explain.
Answer: Yes. if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ then $(a, b) \sim (e, f)$

(d) Is $\sim$ an equivalence relation? Explain.
Answer: Yes. It is reflexive, symmetric, and transitive

11. Let (a, b) and (c, d) be ordered pairs of integers. Show that

$$(a, b) \sim (c, d) \text{ if } a + d = b + c$$

is an equivalence relation.

Answer: For all ordered pairs (a, b), $a + b = b + a$. For all ordered pairs (a, b) and (c, d), if $a + d = b + c$, then $c + b = d + a$. For all ordered pairs (a, b), (c, d), and (e, f) if $a + d = b + c$ and $c + f = d + e$, then $a + f = a + (d + e - c) = b + e$.

12. For the set of all functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that f is a constant multiple of g, that is, there is some non-zero constant $k \neq 0$ such that $f(x) = kg(x)$ for all $x \in \mathbb{R}$. Show that $\sim$ is an equivalence relation.

Answer: For all functions f, $f(x) = 1 \cdot f(x)$, that is $\sim$ is reflexive. For all functions f and g, If $f(x) = k \cdot g(x)$, then $g(x) = \frac{1}{k} \cdot f(x)$, that is $\sim$ is symmetric. For all functions f, g, and h If $f(x) = k \cdot g(x)$ and $g(x) = j \cdot h(x)$ then $f(x) = jk \cdot h(x)$, that is $\sim$ is transitive.

13. Is approximately equal 10 in the first section on problems on relations an equivalence relation? Explain.

Answer: No, Approximately equal is not transitive.

Orders

1. (Know that $\leq$ is a partial order.) Is less than or equal ($a \leq b$) on the real numbers a partial order? Explain your answer.

Answer: Yes, less than or equal is reflexive, antisymmetric, and transitive.

2. (Know that $|$ is a partial order.) Is divides ($a \mid b$) on the natural numbers a partial order? Explain your answer.

Answer: Yes, divides is reflexive, antisymmetric, and transitive.

3. (Know that $\subseteq$ is a partial order.) Is subset ($\mathbb{X} \subseteq \mathbb{Y}$) on power set of a set a partial order? Explain your answer.

Answer: Yes, subset is reflexive, antisymmetric, and transitive.

4. (Be able to determine if a relation is a partial order.) Are the following relations partial orders?
   (a) On the set of ordered pairs of natural numbers define $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.
   Answer: $(a, b) \leq (a, b)$ because $a \leq a$ and $b \leq b$. If $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$, then $a = c$ and $b = d$. If $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$, then $a \leq c \leq e$ and $b \leq d \leq f$.
   (b) Let a and b be bit strings of length n. Say $a \leq b$ if $a = a \wedge b$ where the AND operator is computed bit-wise. For example, on $\mathbb{B}^4$, $a = 0101 \leq 1101 = b$.
   Answer: $a = a \wedge a$. If $a \leq b$ and $b \leq a$, then $a = a \wedge b$ and $b = b \wedge a$, therefore $a = b$. If $a \leq b$ and $b \leq c$, then $a = a \wedge b$ and $b = b \wedge c$, therefore $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$.
   (c) Let $\vec{R}$ and $\vec{S}$ be sequences of natural numbers. Say that sequence $\vec{R}$ precedes $\vec{S}$ and write $\vec{R} \leq \vec{S}$ if $r_i \leq s_i$ for all $i = 0, 1, 2, \ldots$ As an example, $\vec{R} = \langle 0, 1, 2, 3, \ldots \rangle$ precedes $\vec{S} = \langle 1, 2, 4, 8, \ldots \rangle$ because $i \leq 2^i$ for all natural numbers i. Show that precedes is a partial order.
   (d) (Be able to determine if a relation is a partial order.) Is approximately equal from problem 10 in the first section on problems on relations a partial order? Explain.
   Answer: No. approximately equal is not antisymmetric and not transitive.
5. (Be able to construct an order.) The U. S. Post Office can increase its efficiency by sorting letters based on width and height. Define a partial order on a set of ordered pairs (w, h) that describe the width and height of a letter.
   Answer: One possible order is $(w_0, h_0) \leq (w_1, h_1)$ if $w_0 \leq w_1$ and $h_0 \leq h_1$. Note that
   (a) $(w_0, h_0) \leq (w_0, h_0)$
   (b) If $(w_0, h_0) \leq (w_1, h_1)$ and $(w_1, h_1) \leq (w_0, h_0)$, then $w_0 = w_1$ and $h_0 = h_1$.
6. (Be able to construct partial orders.) The U. S. Post Office also ships packages that have a width, height, and depth. Define a partial order on ordered triples (w, h, d) to help the USPS sort packages.
7. A total order $\sim$ is antisymmetric, transitive, and total in that $(\forall x, y)((x \sim y) \vee (y \sim x))$. Which of the following are total orders?
   (a) Less than
   Answer: Less than is not a total order.
   (b) Less than or equal
   Answer: Less than or equal is a total order.
   (c) Divides
   Answer: Divides is not a total order.
   (d) Subset
   Answer: Subset is not a total order.
8. A well-ordered set $\mathbb{X}$ is a set with a total order $\sim$ such that every non-empty subset has a least element. Using the less than or equal order, which of the following sets are well-ordered?
   (a) The natural numbers $\mathbb{N}$.
   Answer: The natural numbers are well-ordered.
   (b) The integers $\mathbb{Z}$.
   Answer: The integers are not well-ordered: The negative numbers do not contain a least element.
   (c) The real numbers $\mathbb{R}$.
   Answer: The real numbers are not well-ordered: The open interval $(0, 1)$ does not have a least element.

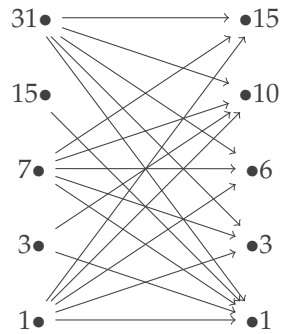
Representation of Relations

Background

Relations can be represented in many ways. Sets of ordered pairs, graphs, and adjacency matrices are common representations.

1. (Be able to draw graphs of relations.) Draw a bipartite graph from the set $\mathbb{X} = \{1, 3, 7, 15, 31\}$ to the set $\mathbb{Y} = \{1, 3, 6, 10, 15\}$ with edges defined by the relatively prime relation, that is, the greatest common denominator of x and y, $\gcd(x, y)$, is equal to 1.

   Answer:



2. (Be able to create an adjacency matrix that describes a relation.) Construct a 5×5 adjacency matrix from the set $\mathbb{X} = \{1, 3, 7, 15, 31\}$ to the set $\mathbb{Y} = \{1, 3, 6, 10, 15\}$ with entries defined by the relatively prime relation, that is, the greatest common denominator of x and y, $\gcd(x, y)$, is equal to 1.

   Answer:

$$\left[\begin{array}{c|ccccc} & 1 & 3 & 6 & 10 & 15 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 7 & 1 & 1 & 1 & 1 & 1 \\ 15 & 1 & 0 & 0 & 0 & 0 \\ 31 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

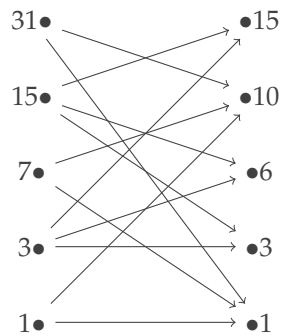
3. (Be able to describe a relation as a set of ordered pairs.) Construct a set of ordered pairs that represents the relation relatively prime from the set $\mathbb{X} = \{1, 3, 7, 15, 31\}$ to the set $\mathbb{Y} = \{1, 3, 6, 10, 15\}$.

   Answer:

$$\begin{aligned} \{(1,1),\ (1,3),\ (1,6),\ (1,10),\ (1,15),\\ (3,1),\ (3,10),\\ (7,1),\ (7,3),\ (7,6),\ (7,10),\ (7,15),\\ (15,1),\\ (31,1),\ (31,3),\ (31,6),\ (31,10),\ (31,15)\} \end{aligned}$$

4. (Know that a relation can be represented as a graph from $\mathbb{X}$ to $\mathbb{Y}$.) A relation from $\mathbb{X}$ to $\mathbb{Y}$ can be represented as a bipartite graph from $\mathbb{X}$ to $\mathbb{Y}$.

   Answer: This is `True`. A bipartite graph only has edges from elements (nodes) in $\mathbb{X}$ to elements (nodes) in $\mathbb{Y}$. For example, the congruence mod 3 relation from $\mathbb{X} = \{1, 3, 7, 15, 31\}$ to the set $\mathbb{Y} = \{1, 3, 6, 10, 15\}$ can be represented by the graph



5. (Know that a relation can be represented as an adjacency matrix.) A relation from $\mathbb{X}$ to $\mathbb{Y}$ can be represented as a adjacency matrix with rows labeled by elements in $\mathbb{X}$ and columns labeled by elements in $\mathbb{Y}$..

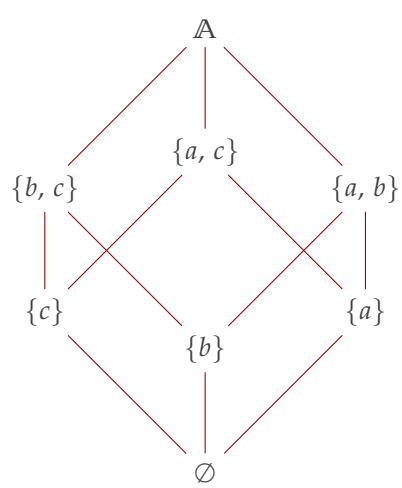
   Answer: This is `True`.

6. (Know that a relation can be represented as a set of ordered pairs.) A relation from $\mathbb{X}$ to $\mathbb{Y}$ can be represented as a a set of ordered pairs.

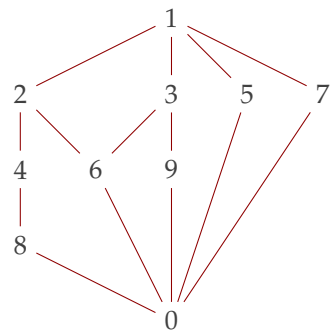
   Answer: This is `True`.

7. (Be able to draw Hasse diagrams of partial orders.) Partial orders are drawn using Hasse diagrams: Graphs where reflexive and transitive edges are not drawn. Draw Hasse diagrams for the following partial orders.

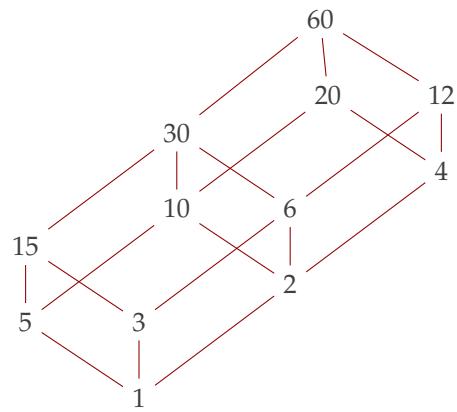
   (a) Subset on the power set of $\{a, b, c\}$.
   Answer:



   (b) Divides on the digits $\{0, 1, 2, \ldots, 9\}$.
   Answer:



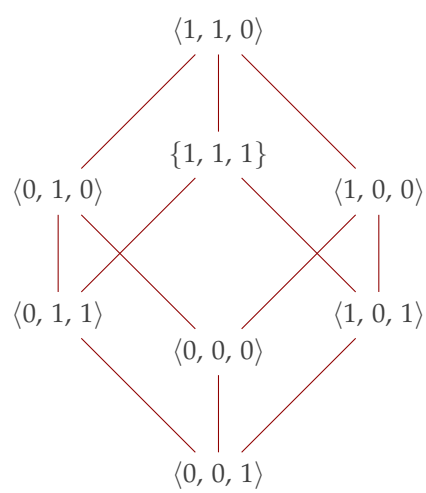
(c) Divides on the divisors of 60.
Answer:



(d) Using the precedes relation on sequence from 4c Draw a graph showing the order of the 8 finite sequences

$$\langle 0, 0, 0\rangle \quad \langle 0, 0, 1\rangle \quad \langle 0, 1, 0\rangle \quad \langle 0, 1, 1\rangle$$
$$\langle 1, 0, 0\rangle \quad \langle 1, 0, 1\rangle \quad \langle 1, 1, 0\rangle \quad \langle 1, 1, 1\rangle$$

Answer:



Counting Relations

Background

There are 2^{nm} relations from a set $\mathbb{X}$ to a set $\mathbb{Y}$ where $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$

1. (Be able to count the number of entries in a matrix.) Let M be an $n \times n$ matrix.

   (a) How many entries are in M?
   Answer: There are n rows with n entries per row: There are $n \times n = n^2$ entries. This can be thought of as summing a 1 for each column $j = 0, 1, \ldots, (n-1)$ and for each row $i = 0, 1, \ldots, (n-1)$

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1$$

   (b) How many entries are there along the main diagonal of M?
   Answer: There are n entries along the diagonal of an $n \times n$ matrix.

   (c) How many entries are there above the main diagonal of M?
   Answer: You can count the number of entries as 1 in the corner, 2 in the next diagonal down, 3 in the next diagonal, all the way to the diagonal above the main diagonal.

$$\begin{bmatrix} x & \frac{(n-1)(n-2)}{2}+1 & \frac{(n-2)(n-3)}{2}+1 & \cdots & 7 & 4 & 2 & 1 \\ x & x & \frac{(n-1)(n-2)}{2}+2 & \cdots & \vdots & 8 & 5 & 3 \\ x & x & x & \ddots & \vdots & \vdots & 9 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & 10 \\ x & x & x & \vdots & x & x & \frac{n(n-1)}{2}-1 & \vdots \\ x & x & x & \cdots & x & x & x & \frac{n(n-1)}{2} \end{bmatrix}$$

   The triangle number

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

   computes the count of entries above the main diagonal of an $n \times n$ matrix.

   (d) How many entries are there on or above the main diagonal of M?
   Answer: There are

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

   on or above the main diagonal.

2. (Be able to count the number of adjacency (Boolean) matrices.) If each entry in an $m \times n$ matrix M can be either a 0 or a 1, how many different matrices are there?

   Answer: There are 2^{mn} different matrices.

3. ⚠ (Be able to count the number of relations on a set.) How many relations can be defined on the following sets?

   (a) The sets of bits $\mathbb{B} = \{0, 1\}$.

Answer: A relation from $\mathbb{B}$ to $\mathbb{B}$ can be represented as a 2×2 adjacency matrix

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} x & y \\ z & w \end{pmatrix} \end{array}$$

where each entry x, y, z and w can be either a 0 or a 1. For example, $x = 0$ is 0 is not related to 0 and $y = 1$ if 0 is related to 1. Since there are 4 entries and there are 2 possible values for each entry, there are 2^4 different matrices and so $2^4 = 16$ different relations. Another explanation is: There are 4 ordered pairs in the set $\mathbb{B} \times \mathbb{B}$.

$$\mathbb{B} \times \mathbb{B} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

Every subset of $\mathbb{B} \times \mathbb{B}$ is a relation. There are 2^4 subsets of a 4 element set, so there are $2^4 = 16$ relations on $\mathbb{B}$.

(b) The sets of digits $\mathbb{D} = \{0, 1, \ldots, 9\}$.
Answer: A relation from $\mathbb{D}$ to $\mathbb{D}$ can be represented as a 10×10 adjacency matrix

$$\begin{array}{cc} & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} & a_{0,5} & a_{0,6} & a_{0,7} & a_{0,8} & a_{0,9} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} \\ a_{2,0} & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} \\ a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & a_{4,9} \\ a_{5,0} & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & a_{5,9} \\ a_{6,0} & a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & a_{6,9} \\ a_{7,0} & a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} & a_{7,8} & a_{7,9} \\ a_{8,0} & a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & a_{8,8} & a_{8,9} \\ a_{9,0} & a_{9,1} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & a_{9,8} & a_{9,9} \end{pmatrix} \end{array}$$

where each entry $a_{i,j}$ can be either a 0 or a 1. Since there are 100 entries and there are 2 possible values for each entry, there are 2^{100} different matrices and so 2^{100} different relations. Another explanation is: There are 100 ordered pairs in the set $\mathbb{D} \times \mathbb{D}$. Every subset of $\mathbb{D} \times \mathbb{D}$ is a relation. There are 2^{100} subsets of $\mathbb{D} \times \mathbb{D}$, so there are $2^{100} \approx 10^{30}$ relations on $\mathbb{D}$.

(c) From the bits to the digits.
Answer: A relation from $\mathbb{B}$ to $\mathbb{D}$ can be represented as a 2×10 adjacency matrix

$$\begin{array}{cc} & \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & a_{0,4} & a_{0,5} & a_{0,6} & a_{0,7} & a_{0,8} & a_{0,9} \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} \end{pmatrix} \end{array}$$

where each entry $a_{i,j}$, can be either a 0 or a 1. Since there are 20 entries and there are 2 possible values for each entry, there are 2^{20} different matrices and so 2^{20} different relations. Another explanation is: There are 20 ordered pairs in the set $\mathbb{B} \times \mathbb{D}$. Every subset of $\mathbb{B} \times \mathbb{D}$ is a relation. There are 2^{20} subsets of $\mathbb{B} \times \mathbb{D}$, so there are $2^{20} \approx 1$ million relations from $\mathbb{B}$ to $\mathbb{D}$.

(d) From $\mathbb{B}^n$, the bit strings of length n, to the bits $\mathbb{B}$.
Answer: There are 2^n bit strings of length n. There are $2^n \times 2 = 2^{n+1}$ ordered pairs in the set $\mathbb{B}^n \times \mathbb{B}$. Every subset of $\mathbb{B}^n \times \mathbb{B}$ is a relation. There are $2^{2^{n+1}}$ subsets of $\mathbb{B}^n \times \mathbb{B}$, so there are $2^{2^{n+1}}$ relations from $\mathbb{B}^n$ to $\mathbb{B}$. Do not be confused, as I was when I re-read this. Every function is a relation, so in general, there are fewer functions than relations. There are 2^{2^n} functions from $\mathbb{B}^n$ to $\mathbb{B}$ and there are $2^{2^{n+1}}$ relations.

4. (Know the number of relations that can be defined on a set.) How many relations can be defined on an n-element set?
Answer: There are n^2 entries in an $n \times n$ matrix. In an adjacency matrix each entry can either be a 0 or a 1 (two choices). Therefore there are 2^{n^2} different adjacency matrices and there are 2^{n^2} relations. Another explanation is: The Cartesian product $\mathbb{X} \times \mathbb{X}$ has cardinality n^2. There are 2^{n^2} subset of $\mathbb{X} \times \mathbb{X}$, and each is a relation on $\mathbb{X}$.

5. (Know the number of relations that can be defined from one set to another.) How many relations can be defined from $\mathbb{X}$ to $\mathbb{Y}$ when $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$?
Answer: There are nm entries in an $n \times m$ matrix. In an adjacency matrix each entry can either be a 0 or a 1 (two choices). Therefore there are 2^{nm} different adjacency matrices and there are 2^{nm} relations. Another explanation is: The Cartesian product $\mathbb{X} \times \mathbb{Y}$ has cardinality nm. There are 2^{nm} subset of $\mathbb{X} \times \mathbb{Y}$, and each is a relation and each is a relation from $\mathbb{X}$ to $\mathbb{Y}$.

6. (Know a formula for computing the number of reflexive relations.) Given that there are

$$2^{(\text{count of off-diagonal elements})}$$

reflexive relations on a set, how many reflexive relations are there for $n = 0, 1, 2, 3, 4$ and n?
Answer: There are $2(1 + 2 + 3 + \cdots + (n-1)) = n(n-1)$ off-diagonal elements in an $n \times n$ adjacency matrix. Therefore there are $2^{0(0-1)} = 1$, $2^{1(1-1)} = 1$, $2^{2(2-1)} = 4$, $2^{3(3-1)} = 64$, $2^{4(4-1)} = 4096$ and $2^{n(n-1)}$ reflexive relations on 0, 1, 2, 3, 4 and n element sets.

7. (Know a formula for computing the number of symmetric relations.) Given that there are

$$f(n) = 2^n 2^{n(n-1)/2} = 2^{n(n+1)/2} = \sqrt{2^{n(n+1)}}$$

symmetric relations on a set with cardinality n. Compute the value of $f(n)$ for $n = 0, 1, 2, 3$ and 4.
Answer: There is 1 symmetric relation on the empty set: The empty relation is symmetric. There are 2 symmetric relation on the singleton set $\{a\}$: The empty relation and $\{(a, a)\}$. There are 8 symmetric relation on the set $\{a, b\}$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{Empty relation} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \{(a, b), (b, a)\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \{(a, a)\} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \{(a, a), (a, b), (b, a)\}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \{(b, b)\} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \{(a, b), (b, a), (b, b)\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \{(a, a), (b, b)\} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \{(a, a), (a, b), (b, a), (b, b)\}$$

There are 64 symmetric relation on the set $\{a, b, c\}$. There are 1024 symmetric relation on the set $\{a, b, c, d\}$.

8. (Be able to count the reflexive relations defined on a set.) How many reflexive relations can be defined on the following sets?

   (a) $\mathbb{B}$

   Answer: A reflexive adjacency matrix on the bits has the form

$$\begin{pmatrix} 1 & b_{01} \\ b_{10} & 1 \end{pmatrix}$$

   where b_{01} and b_{10} can be one of two values: 0 or 1. Therefore there are $2^2 = 4$ reflexive relations on the bits.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   (b) $\mathbb{D}$

   Answer: A reflexive relation has 1 along the main diagonal. A 10 matrix has $1 + 2 + 3 + \cdots + 9 = 9(10)/2 = 45$ entries below the main diagonal and 45 entries above the main diagonal for a total of 90 off-diagonal entries. Each of these entries can be one of two values: 0 or 1. Therefore there are 2^{90} reflexive relation.

9. (Be able to count the symmetric relations defined on a set.) How many symmetric relations can be defined on the following sets?

   (a) $\mathbb{B}$

   Answer: A symmetric adjacency matrix on the bits has the form

$$\begin{pmatrix} b_{0,0} & b \\ b & b_{1,1} \end{pmatrix}$$

   where $b_{0,0}$ and $b_{1,1}$ can be one of two values: 0 or 1 and the off-diagonal bits are equal. Therefore there are $2^2 2 = 2^3 = 8$ symmetric relations on the bits.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   (b) $\mathbb{D}$

   Answer: A symmetric relation has equal values at off-diagonal entries $b_{i,æ}$, $j \neq j$. For a 10×10 matrix there are $1 + 2 + 3 + \cdots + 9 = 9(10)/2 = 45$ entries below the main diagonal and 10 entries on the diagonal for a total of 55 entries on or below the diagonal. Each of these entries can be one of two values: 0 or 1. Therefore there are 2^{55} symmetric relation.

10. Let $m = |\mathbb{X}|$ to $n = |\mathbb{Y}|$ be the number of elements in finite sets $\mathbb{X}$ and $\mathbb{Y}$. How many relations can be defined from $\mathbb{X}$ to $\mathbb{Y}$?

    Answer: There are 2^{mn} different relations.

11. How many reflexive relations can be defined on a set $\mathbb{X}$ with cardinality $|\mathbb{X}| = n$?

    Answer: To be reflexive the main diagonal of the adjacency matrix for the relation must contain only 1's. The off-diagonal elements can be either 0 or 1. There are $1 + 2 + 3 + \cdots + (n-1) = n(n-1)/2$ off-diagonal entries below the main diagonal, therefore there are $n(n-1)/2$ off-diagonal entries. There are $2^{n(n-1)}$ reflexive relations.

12. How many symmetric relations can be defined on a set $\mathbb{X}$ with cardinality $|\mathbb{X}| = n$?

    Answer: To be symmetric the `True/False` value in row i column j must match the `True/False` value in row j column i. There are $1 + 2 + 3 + \cdots + (n-1) + n = n(n+1)/2$ entries on and below the main diagonal. There are $2^{n(n+1)/2} = \sqrt{2}^{n(n+1)}$ symmetric relations.

Problems on Functions

Background

A function $f : \mathbb{X} \to \mathbb{Y}$ is a relation with the property that if $f(x) = y_1$ and $f(x) = y_2$, then $y_1 = y_2$. That is, a function is a DETERMINISTIC relation: Given input x the output y is uniquely determined.

1. ⚠ (Know basic terminology and concepts about functions.) Let $f : \mathbb{X} \to \mathbb{Y}$ be a function. Answer the following `True` or `False`. Explain your answer.

   (a) The set $\mathbb{X}$ is the domain of function f.
   Answer: This is `True`. We will assume that all functions f are "total," meaning for all $x \in \mathbb{X}$, $f(x)$ is defined and an element in $\mathbb{Y}$.

   (b) The set $\mathbb{X}$ is the range (or image) of function f.
   Answer: This is `False`. The range of f is the set $f(\mathbb{X}) = \{y : (\exists x \in \mathbb{X})(y = f(x))\}$.

   (c) The set $\mathbb{X}$ is the co-domain of function f.
   Answer: This is `False`. The co-domain of f is the set $\mathbb{Y}$. Notice the range of f is a subset of $\mathbb{Y}$.

   (d) The set $\mathbb{Y}$ is the range of function f.
   Answer: This is generally `False`. It will be `True` when f is an onto function. The range of f is the set $f(\mathbb{X}) = \{y : (\exists x \in \mathbb{X})(y = f(x))\}$.

   (e) The set $\mathbb{Y}$ is the co-domain of function f.
   Answer: This is `True`.

   (f) The set $\mathbb{Y}$ is the domain of function f.
   Answer: This is `False`. The domain of f is the set $\mathbb{X}$.

2. (Be able to determine if a set of ordered pairs is or is not a function.) Let $\mathbb{A} = \{a, b, c\}$ and $\mathbb{B} = \{0, 1\}$. Which of the following are functions from $\mathbb{A}$ to $\mathbb{B}$.

   (a) $\{(a, 0), (b, 1), (c, 0)\}$
   Answer: This is a function: Each domain value maps to one and only one co-domain value.

   (b) $\{(a, 0), (a, 1), (c, 0)\}$
   Answer: This is not a function: a maps to two different values.

   (c) $\{(a, b), (b, a), (c, c)\}$
   Answer: This is not a function from $\mathbb{A}$ to $\mathbb{B}$, but it is a function from $\mathbb{A}$ to $\mathbb{A}$.

   (d) $\{(a, 1), (b, 1)\}$
   Answer: This is a "partial" function: It is not defined on domain element c.

3. Consider this algorithm for constructing a function f from $\mathbb{X}$ to $\mathbb{Y}$.

For each element $x \in \mathbb{X}$ choose an element $y \in \mathbb{Y}$ and place the ordered pair (x, y) in f.

(a) If the cardinality of $\mathbb{X}$ is 5, how many times does the algorithm choose an x from $\mathbb{X}$?
Answer: There are 5 different choices.

(b) If the cardinality of $\mathbb{Y}$ is 9, how many choice of y are there for each x?
Answer: There are 9 choices of y.

(c) `True` or `False`: If the $|\mathbb{X}| = 5$ and $|\mathbb{Y}| = 9$, then there are 5^9 functions f from $\mathbb{X}$ to $\mathbb{Y}$. Explain your answer.
Answer: This is `False`. There are 9^5 functions: Each of the 5 domain values can map to one of 9 co-domain values.

(d) `True` or `False`: If the $|\mathbb{X}| = 5$ and $|\mathbb{Y}| = 9$ then there are 45 functions f from $\mathbb{X}$ to $\mathbb{Y}$. Explain your answer.
Answer: This is `False`. There are 9^5 functions: Each of the 5 domain values can map to one of 9 co-domain values.

(e) `True` or `False`: If the $|\mathbb{X}| = 5$ and $|\mathbb{Y}| = 9$ then there are 9^5 functions f from $\mathbb{X}$ to $\mathbb{Y}$. Explain your answer.
Answer: This is `True`. Each of the 5 domain values can map to one of 9 co-domain values.

(f) If $|\mathbb{X}| = n$ and $|\mathbb{Y}| = m$ how many functions f can be defined from $\mathbb{X}$ to $\mathbb{Y}$?
Answer: There are $m^n = |\mathbb{Y}|^{|\mathbb{X}|}$ functions.

4. Answer the following `True` or `False`. Explain your answer.

(a) The domain of a function of one Boolean variable is $\{0, 1\}$.
Answer: This is `True`.

(b) The domain of a function of two Boolean variables is $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
Answer: This is `True`.

(c) The co-domain of a Boolean function is $\{0, 1\}$.
Answer: This is `True`.

5. Let $f : \mathbb{X} \to \mathbb{Y}$ be a function. Answer the following `True` or `False`. Explain your answer.

(a) If for every $y \in \mathbb{Y}$ there is an $x \in \mathbb{X}$ such that $f(x) = y$, then f is one-to-one.
Answer: This is `False`. The statement says that f is onto.

(b) If for every $y \in \mathbb{Y}$ there is an $x \in \mathbb{X}$ such that $f(x) = y$, then f is onto.
Answer: This is `True`.

(c) If for every pair $x_1, x_2 \in \mathbb{X}$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, then f is one-to-one.
Answer: This is `True`.

(d) If for every pair $x_1, x_2 \in \mathbb{X}$, $x_1 \neq x_2$ implies $f(x_1) = f(x_2)$, then f is onto.
Answer: This is `False`. The statement says that f is a constant function.

6. Define a function f that maps the two's complement numbers from 0 to $2^n - 1$ onto the signed integers from -2^{n-1} to $2^{n-1} - 1$.

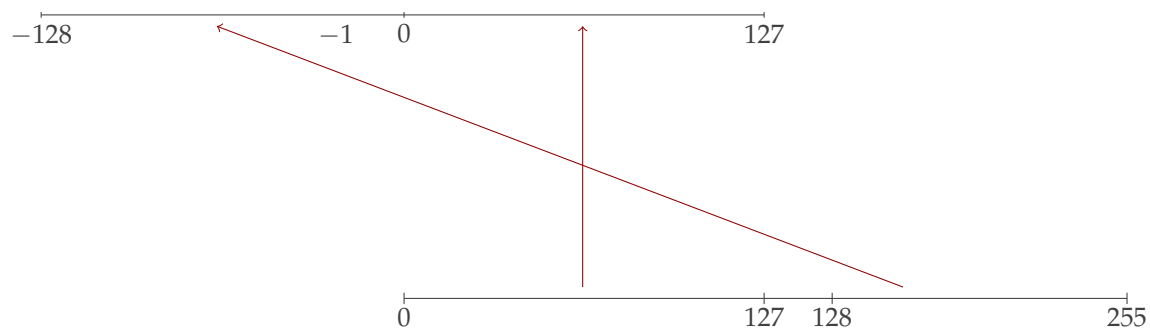
Answer: Two's complement notation maps n bit binary numbers that start with 0 to themselves. That is, the function f will map values k from 0 to $2^{n-1} - 1$ to themselves.

$$f(k) = k \quad \text{if } 0 \leq k < 2^{n-1}$$

(The natural number $2^{n-1} - 1$, written in binary, is a 0 followed by $n - 1$ 1's.) On the other hand, values k from 2^{n-1} to $2^n - 1$ are shifted left along the number line by an amount 2^n. That is, the function f will maps values k from 2^{n-1} to $2^n - 1$ to $k - 2^n$.

$$f(k) = k - 2^n \quad \text{if } 2^{n-1} \leq k < 2^n$$

Here's a sketch that maps the interval 0 to $2^8 - 1 = 255$ to the interval -128 to -1.

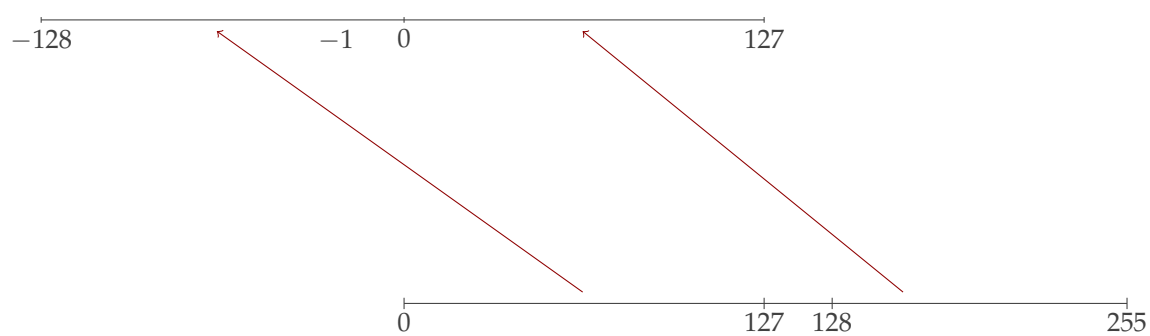


7. Define a function that maps the biased numbers from 0 to $2^n - 1$ onto the signed integers from -2^{n-1} to $2^{n-1} - 1$.

   Answer: Biased notation maps values between 0 an $2^n = 1$ to signed integers by subtracting a bias b, which for this problem is $b = 2^{n-1}$. That is, the function f will maps values k from 0 to $2^n - 1$ to $k - 2^{n-1}$.

$$f(k) = k - 2^{n-1} \quad \text{if } 0 \leq k < 2^n$$

   Here's a sketch that maps the interval 0 to $2^8 - 1 = 255$ to the interval -128 to 127.



Counting Functions

Background

There are $|\mathbb{Y}|^{|\mathbb{X}|}$ (total) functions from $\mathbb{X}$ to $\mathbb{Y}$.

1. ⚠ (Be able to count functions in small examples.) How many (total) functions can be defined given the following domains and co-domains.

   (a) $f : \mathbb{B} \to \{0\}$
   Answer: There is only one function $f(0) = 0$, $f(1) = 0$. There are $1^2 = 1$ functions.

(b) $f : \mathbb{B} \to \mathbb{B}$
Answer: There four functions
i. $f(0) = 0$, $f(1) = 0$
ii. $f(0) = 0$, $f(1) = 1$
iii. $f(0) = 1$, $f(1) = 0$
iv. $f(0) = 1$, $f(1) = 1$
There are $2^2 = 4$ functions.

(c) $f : \{0, 1, 2\} \to \mathbb{B}$
Answer: There eight functions
i. $f(0) = 0$, $f(1) = 0$, $f(2) = 0$
ii. $f(0) = 0$, $f(1) = 1$, $f(2) = 0$
iii. $f(0) = 1$, $f(1) = 0$, $f(2) = 0$
iv. $f(0) = 1$, $f(1) = 1$, $f(2) = 0$
v. $f(0) = 0$, $f(1) = 0$, $f(2) = 1$
vi. $f(0) = 0$, $f(1) = 1$, $f(2) = 1$
vii. $f(0) = 1$, $f(1) = 0$, $f(2) = 1$
viii. $f(0) = 1$, $f(1) = 1$, $f(2) = 1$
There are $2^3 = 8$ functions.

2. ⚠ (Be able to count Boolean functions.) Boolean functions, those that have Boolean values as input and output are an important class of functions.

(a) How many Boolean functions $B(\mathrm{P})$ in one variable are there?
Answer: There are $4 = 2^{2^1}$ Boolean functions:

$$B(\mathrm{P}) = \texttt{False} \qquad B(\mathrm{P}) = \mathrm{P} \qquad B(\mathrm{P}) = \neg\mathrm{P} \qquad B(\mathrm{P}) = \texttt{True}$$

(b) How many Boolean functions $B(\mathrm{P}, \mathrm{Q})$ in two Boolean variables are there?
Answer: There are $16 = 2^{2^2}$ Boolean functions.

$$\begin{array}{llll}
B(\mathrm{P}, \mathrm{Q}) = \texttt{False} & B(\mathrm{P}, \mathrm{Q}) = \mathrm{P} & B(\mathrm{P}, \mathrm{Q}) = \neg\mathrm{P} & B(\mathrm{P}, \mathrm{Q}) = \texttt{True} \\
B(\mathrm{P}, \mathrm{Q}) = \mathrm{P} \wedge \mathrm{Q} & B(\mathrm{P}, \mathrm{Q}) = \mathrm{Q} & B(\mathrm{P}, \mathrm{Q}) = \neg\mathrm{Q} & B(\mathrm{P}, \mathrm{Q}) = \mathrm{P} \vee \mathrm{Q} \\
B(\mathrm{P}, \mathrm{Q}) = \mathrm{P} \to \mathrm{Q} & B(\mathrm{P}, \mathrm{Q}) = \neg(\mathrm{P} \to \mathrm{Q}) & B(\mathrm{P}, \mathrm{Q}) = \mathrm{Q} \to \mathrm{P} & B(\mathrm{P}, \mathrm{Q}) = \neg(\mathrm{Q} \to \mathrm{P}) \\
B(\mathrm{P}, \mathrm{Q}) = \neg(\mathrm{P} \vee \mathrm{Q}) & B(\mathrm{P}, \mathrm{Q}) = \neg(\mathrm{P} \wedge \mathrm{Q}) & B(\mathrm{P}, \mathrm{Q}) = \mathrm{P} \equiv \mathrm{Q} & B(\mathrm{P}, \mathrm{Q}) = \mathrm{Q} \oplus \mathrm{P}
\end{array}$$

A useful way to list these 16 functions is by truth table. I use the binary value of each output column

as a check-sum to know that each of the 16 Boolean functions are listed.

Input		Boolean Functions							
P	Q	0	$P \wedge Q$	$\neg(P \to Q)$	P	$\neg(Q \to P)$	Q	$P \oplus Q$	$P \vee Q$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1
Binary		0	1	2	3	4	5	6	7

Input		Boolean Functions							
P	Q	$\neg(P \vee Q)$	$\equiv$	$\neg Q$	$Q \to P$	$\neg P$	$P \to Q$	$\neg(P \wedge Q)$	1
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1
Binary		8	9	10	11	12	13	14	15

(c) How many Boolean functions $B(P, Q, R)$ in three Boolean variables are there?
Answer: There are $256 = 2^8 = 2^{2^3}$ Boolean functions: Too many to list.

(d) How many Boolean functions $B(P_0, P_1, \ldots, P_{n-1})$ in n Boolean variables are there?
Answer: There are 2^{2^n} Boolean functions in n variables.

3. (Be able to count functions in the general case of a finite domain and finite co-domain.) How many (total) functions are there from a 5 element domain to a 7 element co-domain?

Answer: There are 7^5 functions. One way to visualize this is to represent a function as an adjacency matrix. Here's an example.

$$\left[\begin{array}{c|ccccccc} & y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ \hline x_0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ x_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since the matrix represents a function, there can be at most one 1 in every row: A function maps each input value to only one output. Since the function is total, there is a 1 in each row. For each row, there are 7 choices of where to put the 1. Therefore, there are 7^5 different functions.

4. (Be able to count functions) How many (total) functions are there from a m element domain to a n element co-domain?

Answer: There are n^m (total) functions. A function $f : \mathbb{X} \to \mathbb{Y}$ where $|\mathbb{X}| = m$ and $|\mathbb{Y}| = n$ can be thought of as an $m \times n$ adjacency matrix where each row contains one 1 with the other entries being 0. For each row there are n choices of where to place the 1. There are m rows, so there are a total of n^m different matrices that represent functions.

5. (Be able to count partial functions.) How many partial functions are there from a m element domain to a n element co-domain?
   Answer: For each $k = 0, 1, 2, \ldots, m$ choose k elements to map. For each choice of k elements there are n^k choices for the map. Therefore, by the binomial theorem, there are

$$\sum_{k=0}^{m} \binom{m}{k} n^k = (n+1)^m$$

   different partial functions.

Problems on Polynomials

Background

Polynomials, sums of powers of x weighted by coefficients, are fundamental functions.

1. (Be able to identify the degree of a polynomial.) Compute the degree of the polynomials
   (a) $p(x) = x^5 + 4x^3 + 2x - 7$
   Answer: The degree is 5, the largest exponent.
   (b) $p(x) = 5x^{10} - 1$
   Answer: The degree is 10, the largest exponent.
   (c) $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$
   Answer: The degree is $n - 1$, the largest exponent.
   (d) $p(x) = \sum_{0 \le k \le n-1} a_k x^k$
   Answer: The degree is $n - 1$, the largest exponent.

2. (Be able to compute the roots (zeros) of a polynomial of small degree.)
   (a) What are the roots of the polynomial equation $5x - 37 = 0$?
   Answer: $x = 37/5$
   (b) What are the roots of the polynomial equation $mx + b = 0$, $m \neq 0$?
   Answer: $x = -b/m$
   (c) What are the roots of the polynomial equation $x^2 - x - 1 = 0$?
   Answer: The two roots are the golden ration

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

   and its conjugate

$$\overline{\varphi} = \frac{1 - \sqrt{5}}{2}$$

   (d) What are the roots of the polynomial equation $ax^2 + bx + c = 0$, $a \neq 0$?
   Answer: The two roots are given by the quadratic equation

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

   and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

   which may be complex numbers.

3. ⚠ (Know the formula for geometric sums.) Show that the following equations are always `True` provided $x \neq 1$.

   (a) $1 = (x-1)/(x-1)$.
   Answer: This is clearly `True` provided $x \neq 1$. The number 1 is the first term of a geometric sum.

   (b) $1 + x = (x^2 - 1)/(x-1)$.
   Answer: Since $x^2 - 1 = (x-1)(x+1)$ formula is `True` provided $x \neq 1$. The sum $1 + x$ is the first two terms of a geometric sum.

   (c) $1 + x + x^2 = (x^3 - 1)/(x-1)$.
   Answer: Since $x^3 - 1 = (x-1)(x^2+x+1)$ formula is `True` provided $x \neq 1$. The sum $1 + x + x^2$ is the first three terms of a geometric sum.

   (d) $1 + x + x^2 + \cdots + x^{n-2} + x^{n-1} = (x^n - 1)/(x-1)$.
   Answer: By mathematical induction, if

$$1 + x + x^2 + \cdots + x^{n-2} = \frac{x^{n-1} - 1}{x - 1}$$

   then

$$\begin{aligned}(1 + x + x^2 + \cdots + x^{n-2}) + x^{n-1} &= \frac{x^{n-1} - 1}{x - 1} + x^{n-1} \\ &= \frac{x^{n-1} - 1}{x - 1} + x^{n-1}\frac{x - 1}{x - 1} \\ &= \frac{x^n - 1}{x - 1}\end{aligned}$$

4. ⚠ (Know the binomial theorem.) Show that the following equations are always `True`.

   (a) $(x+y)^0 = 1$.
   Answer: This is `True`. When $x = -y$ the equation says $0^0 = 1$, which is the elegant answer, but in calculus you'll learn that 0^0 can be interpreted to be 0.

   (b) $(x+y)^1 = x + y$.
   Answer: This is clearly `True`.

   (c) $(x+y)^2 = x^2 + 2x + 1$.
   Answer: $(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$.

   (d) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
   Answer: Since $(x+y)^2 = x^2 + 2xy + y^2$, we have

$$\begin{aligned}(x+y)^3 &= (x+y)^2(x+y) \\ &= (x^2 + 2xy + y^2)(x+y) \\ &= (x^3 + 2x^2y + xy^2) + (x^2y + 2xy^2 + y^3) \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

(e) $(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k$.
Answer: By mathematical induction, if

$$(x+y)^{n-1} = \sum_{k=0}^{k=n-1} \binom{n-1}{k} x^{n-1-k} y^k$$

. then

$$\begin{aligned}
(x+y)^n &= (x+y)^{n-1}(x+y) \\
&= \left(\sum_{k=0}^{k=n-1} \binom{n-1}{k} x^{n-k} y^k\right) + \left(\sum_{k=0}^{k=n-1} \binom{n-1}{k} x^{n-1-k} y^{k+1}\right) \\
&= x^n + \left(\sum_{k=1}^{k=n-1} \binom{n-1}{k} x^{n-k} y^k\right) + \left(\sum_{k=1}^{k=n-1} \binom{n-1}{k-1} x^{n-1} y^k\right) + y^n \\
&= x^n + \left(\sum_{k=1}^{k=n-1} (\binom{n-1}{k} + \binom{n-1}{k-1}) x^{n-k} y^k\right) + y^n \\
&= \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k
\end{aligned}$$

5. ⚠ (Be able to use Horner's rule to compute the value of a polynomial at a given value of its parameter.) Use Horner's rule to evaluate the following polynomials at the given value of x.

(a) $p(x) = 3x^4 - 5x^2 - 16x - 4$ at $x = 3$.
Answer:

Horner's Rule				
3	0	−5	−16	−4
	9	27	66	150
3	9	22	50	$\boxed{146}$

$\therefore\ p(3) = 146$. In this instance, what Horner's rule computes it the quotient and remainder when $3x^4 - 5x^2 - 16x - 4$ is divided by $x - 3$.

$$\frac{3x^4 - 5x^2 - 16x - 4}{x-3} = 3x^3 + 9x^2 + 22x + 50 + \frac{146}{x-3}$$

This use of Horner's rule is called "synthetic division."

(b) $p(x) = 3x^5 - 4x^3 + 3x^2 - 7$ at $x = 2$.
Answer:

Horner's Rule					
3	0	−4	3	0	−7
	6	12	16	38	76
3	6	8	19	38	$\boxed{69}$

$\therefore p(2) = 69$.

(c) $p(x) = -2x^4 - 3x^3 + 2x^2 + 4x - 3$ at $x = -3$.
Answer:

Horner's Rule				
-2	-3	2	4	-3
	6	-9	21	-75
-2	3	-7	25	$\boxed{-78}$

$\therefore p(-3) = -78$.

(d) $p(x) = x^5 - 2x^3 + 3x^2 - 1$ at $x = -2$.
Answer:

Horner's Rule						
1	0	-2	3	0	-1	
		-2	4	-4	2	-4
	1	-2	2	-1	2	$\boxed{-5}$

$\therefore p(2) = -5$

(e) $p(x) = x^4 + x^3 + x^2 + x + 1$ at $x = 2$.
Answer:

Horner's Rule				
1	1	1	1	1
	2	6	14	30
1	3	7	15	$\boxed{31}$

$\therefore p(2) = 31$. (Do you see the relationship with binary numbers?)

(f) $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$ at $x = 2$.
Answer:

Horner's Rule				
1	4	6	4	1
	2	12	36	80
1	6	18	40	$\boxed{81}$

$\therefore p(2) = 81$ (Do you see the relationship to $(2+1)^4$?)

6. (Know that polynomials can be written using basis functions other than the power functions x^k.) Polynomials can be written in the falling (factorial) power basis $x^{\underline{0}} = 1$, $x^{\underline{1}} = x$, $x^{\underline{2}} = x(x-1)$, $x^{\underline{3}} = x(x-1)(x-2), \ldots, x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ as an alternative to the standard basis 1, x, x^2, $x^3, \ldots, x^n$. Show following equations that relate the power basis to the falling factorial power basis using Stirling numbers of the second kind are `True`.

(a) $x^0 = x^{\underline{0}}$
Answer: $x^0 = 1 = x^{\underline{0}}$.

(b) $x^1 = x^{\underline{1}}$
Answer: $x^1 = x = x^{\underline{1}}$.

(c) $x^2 = x^{\underline{1}} + x^{\underline{2}}$
Answer: $x^2 = x + x(x-1) = x^{\underline{1}} + x^{\underline{2}}$.

(d) $x^3 = x^{\underline{1}} + 3x^{\underline{2}} + x^{\underline{3}}$
Answer: $x^3 = x + 3x(x-1) + x(x-1)(x-2) = x + (3x^2 - 3x) + (x^3 - 3x^2 + 2x)$.

(e) $x^4 = x^{\underline{1}} + 7x^{\underline{2}} + 6x^{\underline{3}} + x^{\underline{4}}$
Answer: $x^3 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3) = x + (7x^2 - 7x) + (6x^3 - 18x^2 + 12x) + (x^4 - 6x^3 + 11x^2 - 6x)$.

7. (Be able to create an efficient algorithm for computing polynomials written using falling factorial powers.) Horner's rule efficiently evaluates a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

written in the standard basis using n additions and n multiplies. Devise an algorithm to evaluate a polynomial

$$a_n x^{\underline{n}} + a_{n-1} x^{\underline{n-1}} + \cdots a_1 x^{\underline{1}} + a_0 x^{\underline{0}}$$

written in the falling power basis. Count the number of additions and multiplications your algorithm uses.

Answer: Consider a small example: Evaluate $a_2 x^{\underline{2}} + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}$. This can be evaluated by the computations

$$\begin{array}{lll} a_2 & a_1 & a_0 \\ & a_2(x-1) & (a_2(x-1)+a_1)x \\ \hline a_2 & a_2(x-1)+a_1 & (a_2(x-1)+a_1)x + a_0 \end{array}$$

In general, the falling factorial polynomial can be evaluated the algorithm:

(a) Let sum $= a_{n-1}$;

(b) For $k = n-2$ down to 0 do $\{\text{sum} = \text{sum} \cdot (x-k) + a_k\}$;

This algorithm requires $n-1$ multiplications and $2n-4$ additions.

Problems on Logarithms and Exponentials

Background

Logarithms simplify calculations by turning products into sums, division into differences, exponentiation into products.

1. (Know basic rules for exponentiation.) Answer `True` or `False`. Explain your answer.

(a) $2^a 2^b = 2^{a+b}$.
Answer: `True`. Multiplying a 2's by the product of b 2's is the same as multiplying $a+b$ 2's together.

(b) $2^a 2^b = 2^{ab}$.
Answer: `False`. As a counterexample, $2^3 2^4 = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2) = 2^7 \neq 2^{12}$. The correct rule is $2^a 2^b = 2^{a+b}$

(c) $2^n 2^n = 2^{2n}$.
Answer: `True`

(d) $2^n 2^n = 2^{n^2}$.
Answer: `False`. As a counterexample, $2^3 2^3 = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^6 \neq 2^{3^2} = 2^9$. The correct rule is $2^n 2^n = 2^{2n}$.

(e) $(2^a)^b = 2^{ab}$.
Answer: `True`

(f) $(2^a)^b = 2^{a^b}$.
Answer: `False`. As a counterexample, $(2^3)^3 = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^9 \neq 2^{3^3} = 2^{27}$. The correct rule is $(2^a)^b = 2^{ab}$.

(g) $(2^n)^n = 2^{n^2}$.
Answer: `True`

(h) $(2^n)^n = 2^{n^n}$.
Answer: `False`

2. (Know basic rules for exponentiation and summations.) Simplify the following products

(a) $2^0 2^1 2^2 \cdots 2^{n-1}$
Answer: The product is $\sqrt{2^{n(n-1)}}$.

(b) $2^1 2^2 2^4 \cdots 2^{2^{n-1}}$
Answer: The product is $2^{2^n - 1}$.

3. ⚠ (Be able to compute the base 2 logarithm at simple values.) Evaluate the logarithm base 2 function $\lg x$ at the given value for x.

(a) $\lg(256)$
Answer: $\lg(256) = 8$ because $2^8 = 256$.

(b) $\lg(0.25)$
Answer: $\lg(0.25) = -2$ because $2^{-2} = 0.25$.

(c) $\lg(0.125)$
Answer: $\lg(0.125) = -3$ because $2^{-3} = 0.125$.

(d) $\lg(1/1024)$
Answer: $\lg(1/1024) = -10$ because $2^{-10} = 1/1024$.

(e) $\lg(\sqrt[3]{16})$
Answer: $\lg(\sqrt[3]{16}) = 4/3$ because $2^{4/3} = \sqrt[3]{16}$.

(f) $\lg(\sqrt[5]{8})$
Answer: $\lg(\sqrt[5]{8}) = 3/5$ because $2^{3/5} = \sqrt[5]{8}$.

(g) $\lg \sqrt[3]{32}$
Answer: The logarithm base 2 of the cube root of 32 is $5/3$ because $2^{5/3} = \sqrt[3]{32}$.

(h) $\log_{16} \sqrt[3]{32}$
Answer: The logarithm base 16 of the cube root of 32 is $5/12$. Note that

$$\log_{16} \sqrt[3]{32} = \frac{1}{3} \log_{16} 32 = \frac{1}{3} \cdot \frac{5}{4} = \frac{5}{12}$$

(i) $\lg(2^{n!})$
Answer: $\lg(2^{n!}) = n!$

4. (Be able to manipulate logarithms and exponentials.) Compute the following values.
   (a) $10^{\lg x}$ for $x = 1, 2, 4, 8$
   Answer:
   $$10^{\lg 1} = 1, \quad 10^{\lg 2} = 10, \quad 10^{\lg 4} = 100, \quad 10^{\lg 8} = 1000$$
   (b) $x^{\lg 10}$ for $x = 1, 2, 4, 8$
   Answer:
   $$1^{\lg 10} = 1, \quad 2^{\lg 10} = 10, \quad 4^{\lg 10} = 100, \quad 8^{\lg 10} = 1000$$
   (c) Is it `True` or `False` that $10^{\lg x} = x^{\lg 10}$? Explain your answer.
   Answer: It is `True`. Notice that If $y = 10^{\lg x}$, then taking the logarithm base 2 of both sides of the equation yields
   $$\lg y = \lg 10^{\lg x} = \lg x \lg 10$$
   Therefore, raising the the sides as powers of 2
   $$y = 2^{\lg y} = 2^{\lg x \lg 10} = x^{\lg 10}$$
   (d) Show that $a^{\log_b x} = x^{\log_b a}$.
   Answer: The log base b of $a^{\log_b x}$ is $\log_b a \log_b x$. The log base b of $x^{\log_b a}$ is $\log_b x \log_b a$. Clearly, the two logarithms are equal and since the logarithm function is one-to-one, the original values, $a^{\log_b x}$ and $x^{\log_b a}$ are equal.

5. ⚠ (Know the relationship between the number of bits needed to name a number n and the base 2 logarithm of n.) In binary notation, how many bits are required to name a natural number n in the following cases.
   (a) $2^3 \leq n < 2^4$?
   Answer: The natural numbers from 8 to 15 are $(1000)_2$ to $(1111)_2$ in binary. That is, $4 = \lfloor \lg n \rfloor + 1$ bits are used.
   (b) In binary notation, how many bits are required to name a natural number n
   Answer: In binary notation, $m = \lfloor \lg n \rfloor + 1$ bits are used.
   (c) `True` or `False`: The logarithm (base 2) function $\lg(x)$ maps the open interval $(0, \infty)$ onto the set of real numbers. Explain your answer.
   Answer: This is `True`. Every value y in the range $(0, \infty)$ is mapped onto by the log function.

6. (Know basic properties of the logarithms.) `True` or `False`: The logarithm (base 2) function $\lg(x)$ is NOT one-to-one. Explain your answer.

   Answer: This is `False`, the logarithm is a one-to-one function.

7. (Be able to approximate the value of logarithms) `True` or `False`: Since $2^{10} = 1024$ is approximately equal to $10^3 = 1000$, the log base 2 of 10 is approximately equal to 3 and $1/3$. Explain your answer.

Answer: This is `True`. Compute the approximations

$$\begin{aligned} 10^3 &\approx 2^{10} \\ \lg(10^3) &\approx \lg(2^{10}) \\ 3\lg(10) &\approx 10 \\ \lg(10) &\approx \frac{10}{3} \end{aligned}$$

8. (Be able to approximate the value of logarithms) Use the fact that $2^7 = 128$ is approximately equal to $5^3 = 125$ to approximate the value of $\log_5(2)$.
   Answer: Since $2^7 \approx 5^3$ we have $\log_5(2^7) = 7\log_5(2) \approx \log_5(5^3) = 3$ or $\log_5(2) \approx 3/7$.

9. (Understand how to convert between logarithm bases) Show that the

$$\log_b(x) = \frac{1}{\log_x(b)}$$

   Answer: If $y = \log_b(x)$, then $b^y = x$. Take the logarithm base x of this last equation to get

$$\log_x(b^y) = y\log_x(b) = 1$$

   so that

$$y = \frac{1}{\log_x(b)} = \log_b(x)$$

10. Show that the

$$\log_{b^n}(x) = \frac{1}{n}\log_b(x)$$

   Answer:

$$\begin{aligned} \frac{1}{n}\log_b(x) &= \frac{1}{n}\frac{1}{\log_x(b)} \\ &= \frac{1}{\log_x(b^n)} \\ &= \log_{b^n}(x) \end{aligned}$$

11. (Understand how to convert between logarithm bases) Show that the

$$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$$

   Answer: If $y = \log_b(x)$, then $b^y = x$. Take the logarithm base c of this last equation to get

$$\log_c(x) = \log_c(b^y) = y\log_c(b)$$

   so that

$$y = \frac{\log_c(x)}{\log_c(b)}$$

12. Show that the

$$\log_{b^n}(x) = \frac{1}{n}\log_b(x)$$

Answer:

$$\begin{aligned}\frac{1}{n}\log_b(x) &= \frac{1}{n}\frac{1}{\log_x(b)} \\ &= \frac{1}{\log_x(b^n)} \\ &= \log_{b^n}(x)\end{aligned}$$

13. (Be able in derive interesting approximations among logarithms) Given that $\ln 2 \approx 0.693147$ and $\log 2 \approx 0.301030$, show that

$$\lg x \approx \ln x + \log x$$

Specifically, that the error is less than 1% in that

$$\left|\frac{\ln x + \log x}{\lg x} - 1\right| < 0.01$$

Answer: Since $\ln x = \ln 2 \lg x$ and $\log x = \log 2 \lg x$ we have

$$\ln x + \log x = \ln 2 \lg x + \log 2 \lg x = (\ln 2 + \log 2)\lg x \approx 0.994177 \lg x$$

or

$$\left|\frac{\ln x + \log x}{\lg x} - 1\right| \approx |0.994177 - 1| < 0.005823$$

14. (Understand the relationship between logarithms of different bases) For each problem below, find a constant $c \in \mathbb{R}$ such that the equation is an identity.

(a) $\lg(x) = c\ln(x)$
Answer: $\lg(x) = \lg(e)\ln(x)$ or $\lg(x) = \ln(x)/\ln(2)$

(b) $\ln(x) = c\lg(x)$
Answer: $\ln(x) = \ln(2)\lg(x)$ or $\ln(x) = \lg(x)/\lg(e)$

(c) $\lg(x) = c\log(x)$
Answer: $\lg(x) = \lg(10)\log(x)$ or $\lg(x) = \log(x)/\log(2)$

(d) $\log(x) = c\lg(x)$
Answer: $\log(x) = \log(2)\lg(x)$ or $\log(x) = \lg(x)/\lg(10)$

(e) $\lg(x) = c\log_b(x)$
Answer: $\lg(x) = \lg(b)\log_b(x)$ or $\lg(x) = \log_b(x)/\log_b(2)$

(f) $\log_b(x) = c\lg(x)$
Answer: $\log_b(x) = \log_b(2)\lg(x)$ or $\log_b(x) = \lg(x)/\lg(b)$

15. (Understand the relationship between logarithms of different bases) Consider the proposition: $\log_a(b) \cdot \log_b(a) = 1$. Is the position `True` or `False`. You may assume a and b are natural numbers greater than or equal to 2. Explain your answer.

16. Use the "sum of logs is log of a product rule" to write the sum

$$\sum_{1 \le k \le n} \lg k$$

as a logarithm that involves a factorial.

Answer: The sum of $\lg k$ from $k = 1$ to n is

$$\sum_{1 \le k \le n} \lg k = \lg n!$$

Interestingly, since $k \le n$ for $k = 1$ to $k = n$, we can write

$$\lg n! \le n \lg n$$

17. (Know some applications of logarithms) A solution's pH measures its acidity or alkalinity, with 7 called neutral. In particular,

$$\text{pH} = -\log_{10}(\text{H}_3\text{O}^+)$$

where H_3O^+ is the concentration of hydronium ions in the solution. The pH scale is logarithmic: When pH increases by 1, alkalinity increases 10-fold. Likewise, when pH decreases by 1, acidity increases 10-fold. Pretend the average pH of sea water will decrease by 0.5 in 100 years. By what factor will the acidity sea water increase?

Answer: Pretend $p = \log c$ is the average pH of sea water today. If $p - 0.5$ is the average pH of sea water in 100 years, then $p - 0.5 = \log ac = \log c + \log a$ where ac is the acid concentration in 100 years and $-0.5 = \log a$ or $a = 1/\sqrt{10} \approx 0.32$. That is the acidity of the seq will increase by about 32%.

18. (Know some applications of logarithms) If a principle of P dollars is invested at an interest rate r compounded n times per year, then the amount after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Pretend $P = \$1$ is invested at a 10% rate compounded quarterly. How many years t will go by before $A = \$10$?

Answer: Setting $10 = (1 + 0.1/4)^{4t} = (1.025)^{4t}$ and solving for t using the common logarithm, find that $1 = 4t \log 1.025$ or $t = 1/(4 \log 1.025) \approx 23.3$ years.

Problems on Integer Functions

Background

Natural numbers and integers are the foundation of discrete mathematics, and it is often required to convert from rationals or real numbers to integers.

1. (Be able to compute the floor (greatest integer) function.) Evaluate the following floors.

(a) $\left\lfloor \sqrt{2} \right\rfloor$

Answer: $\left\lfloor \sqrt{2} \right\rfloor = 1$

(b) $\lfloor \pi \rfloor$

Answer: $\lfloor \pi \rfloor = 3$

(c) $\lfloor e \rfloor$
Answer: $\lfloor e \rfloor = 2$

(d) $\lfloor \varphi \rfloor = \left\lfloor (1+\sqrt{5})/2 \right\rfloor$
Answer: $\lfloor \varphi \rfloor = \left\lfloor (1+\sqrt{5})/2 \right\rfloor = 1$

(e) $\lfloor \overline{\varphi} \rfloor$
Answer: $\lfloor \overline{\varphi} \rfloor = \left\lfloor \frac{1-\sqrt{5}}{2} \right\rfloor = -1$

(f) $\lfloor \gamma \rfloor$
Answer: $\lfloor \gamma \rfloor = 0$

2. (Be able to compute the floor (least integer) function.) Evaluate the following ceilings.

(a) $\left\lceil \sqrt{2} \right\rceil$
Answer: $\left\lceil \sqrt{2} \right\rceil = 2$

(b) $\lceil \pi \rceil$
Answer: $\lceil \pi \rceil = 4$

(c) $\lceil e \rceil$
Answer: $\lceil e \rceil = 3$

(d) $\lceil \phi \rceil = \left\lceil (1+\sqrt{5})/2 \right\rceil$
Answer: $\left\lceil (1+\sqrt{5})/2 \right\rceil = 2$

(e) $\lceil \overline{\varphi} \rceil$
Answer: $\lceil \overline{\varphi} \rceil = 0$

(f) $\lceil \gamma \rceil \approx \lceil 0.5772 \rceil$
Answer: $\lceil \gamma \rceil = 1$

3. (Be able to compute the fractional part of a real number.) Evaluate the following fractional parts rounded to 4 decimal places.

(a) $\left\{ \sqrt{2} \right\}$
Answer: $\left\{ \sqrt{2} \right\} = \sqrt{2} - 1 \approx 0.4142$

(b) $\{\pi\}$
Answer: $\{\pi\} = \pi - 3 \approx 0.1416$

(c) $\{e\}$
Answer: $\{e\} \approx 0.7183$

(d) $\{\phi\} = \left\{ (1+\sqrt{5})/2 \right\}$
Answer: $\left\{ (1+\sqrt{5})/2 \right\} \approx 0.6180$

(e) $\{\overline{\varphi}\} = \left\{ (1-\sqrt{5})/2 \right\}$
Answer: $\left\{ (1-\sqrt{5})/2 \right\} = (1-\sqrt{5})/2 + 1 \approx 0.3120$

(f) $\{\gamma\}$
Answer: $\{\gamma\} \approx 0.5772$

4. (Be able to compute the nearest integer to a real number.) Evaluate the following rounds.

(a) $\left(\sqrt{2} \right)$
Answer: $\left(\sqrt{2} \right) = 1$

(b) (φ)
Answer: $(\varphi) = 2$

(c) $(\overline{\varphi})$
Answer: $(\overline{\varphi}) = -1$

(d) (γ)
Answer: $(\gamma) = 1$

5. (Understand the basic properties of floors and ceilings.) Answer the following `True` or `False`. Explain your answer.

(a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$.
Answer: This proposition is `True`.

(b) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$.
Answer: This proposition is `False`.

(c) $\lceil x \rceil = n$ if and only if $x \leq n < x+1$.
Answer: This proposition is `False`.

(d) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$.
Answer: This proposition is `True`.

(e) $x = \lfloor x \rfloor + \{x\}$, where $\{x\}$ if the fractional part function.
Answer: This proposition is `True`.

(f) If n and m are positive integers, then $\lfloor n/m \rfloor$ is the quotient when n is divided by m.
Answer: This proposition is `True`.

(g) Every integer $n > 0$ can be represented in $\lfloor \lg(n) \rfloor + 1$ bits.
Answer: This proposition is `True`.

(h) Every integer $n > 0$ can be represented in $\lceil \lg(n+1) \rceil$ bits.
Answer: This proposition is mostly `True`, but not at boundaries where $n = 2^k$.

6. (Understand the relationship between the floor of x and the floor of $-x$) Prove that $\lfloor x \rfloor + \lfloor -x \rfloor = 0$ if $x \in \mathbb{Z}$, but $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ if $x \notin \mathbb{Z}$.

Answer: For every real number x there is an integer n such that

$$n \leq x < n + 1 \qquad \text{and} \qquad -1 - n < -x \leq -n$$

If $x \in \mathbb{Z}$, then $x = n$, $-x = -n$, $\lfloor x \rfloor = n$, and $\lfloor -x \rfloor = -n$. Thus $\lfloor x \rfloor + \lfloor -x \rfloor = n - n = 0$. If $x \notin \mathbb{Z}$ then $n = \lfloor x \rfloor$ and $-1 - n = \lfloor -x \rfloor$. Thus $\lfloor x \rfloor + \lfloor -x \rfloor = n + (-1 - n) = -1$.

7. (Understand the relationship between the ceiling of x and the ceiling of $-x$) What can you conclude about the sum of the ceiling of x and the ceiling of $-x$.

Answer: For every real number x there is an integer n such that

$$n - 1 < x \leq n \qquad \text{and} \qquad -n \leq -x < -n + 1$$

If $x \in \mathbb{Z}$, then $x = n$, $-x = -n$, $\lceil x \rceil = n$, and $\lceil -x \rceil = -n$. Thus $\lceil x \rceil + \lceil -x \rceil = n - n = 0$. If $x \notin \mathbb{Z}$ then $n = \lceil x \rceil$ and $-n + 1 = \lceil -x \rceil$. Thus $\lceil x \rceil + \lceil -x \rceil = n + (-n + 1) = 1$.

8. (Understand the relationship between the fractional part of x and the fractional part of $-x$) What can you conclude about the sum of the fractional parts of x and $-x$

$$\{x\} + \{-x\} = ?$$

Answer: For each real number x, $x = \lfloor x \rfloor + \{x\}$ and $-x = \lfloor -x \rfloor + \{-x\}$ If $x \in \mathbb{Z}$, then

$$\begin{aligned} 0 &= -x + x \\ &= (\lfloor x \rfloor + \{x\}) + (\lfloor -x \rfloor + \{-x\}) \\ &= \{x\} + \{-x\} \end{aligned}$$

If $x \notin \mathbb{Z}$, then

$$\begin{aligned} 0 &= -x + x \\ &= (\lfloor x \rfloor + \{x\}) + (\lfloor -x \rfloor + \{-x\}) \\ &= -1 + \{x\} + \{-x\} \end{aligned}$$

Or $\{x\} + \{-x\} = 1$.

9. (Understand the Dirichlet box (pigeonhole) principle.) Answer the following questions about the distribution of objects into boxes.

   (a) If 12 objects are put in 5 boxes some box must contain at least $\lceil 12/5 \rceil = 3$ objects.
   Answer: This proposition is `True`. If every box contain no more than 2 objects there would be only 10 objects.

   (b) If 12 objects are put in 5 boxes some box must contain $\lfloor 12/5 \rfloor = 2$ or fewer objects.
   Answer: This proposition is `True`. If every box contain three or more objects there would be at least 15 objects.

   (c) Prove the Dirichlet box (pigeonhole) principle: If n objects are placed in m boxes then some box contains $\lceil n/m \rceil$ or more objects and some box contains $\lfloor n/m \rfloor$ or less objects.
   Answer: Consider an example: Suppose $n = 15$ and $m = 4$. Then $\lfloor n/m \rfloor = 3$ and $\lceil n/m \rceil = 4$. In this case if all boxes contain less than 4 objects then there are at most 12 objects, and if all boxes contain more than 4 objects then there are 16 objects. If m does not divides n, then $\lfloor n/m \rfloor < n/m < \lceil n/m \rceil$. Therefore $m \lfloor n/m \rfloor < n < m \lceil n/m \rceil$, If m divides n, then $\lfloor n/m \rfloor = \lceil n/m \rceil = n/m$. In this case if all m boxes contain less than $\lceil n/m \rceil = n/m$ objects, then there are less than $m \cdot n/m = n$ objects, and if all m boxes contain more than $\lceil n/m \rceil = n/m$ objects, then there are more than $m \cdot n/m = n$ objects.

10. (Understand and be able to compute the quotient function.) Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, $n \neq 0$ be a non-zero natural number. The quotient function $q(a, n)$ is given by

$$q(a, n) = \left\lfloor \frac{a}{n} \right\rfloor$$

Compute the following quotients.

   (a) $q(7, 2)$
   Answer: $q(7, 2) = \lfloor 7/2 \rfloor = 3$.

   (b) $q(-7, 2)$
   Answer: $q(7, 2) = \lfloor -7/2 \rfloor = -4$.

   (c) $q(5, 6)$
   Answer: $q(5, 6) = \lfloor 5/6 \rfloor = 0$.

   (d) $q(-5, 6)$
   Answer: $q(-5, 6) = \lfloor -5/6 \rfloor = -1$.

   (e) $q(32, 3)$
   Answer: $q(32, 3) = \lfloor 32/3 \rfloor = 10$.

   (f) $q(-32, 3)$
   Answer: $q(-32, 3) = \lfloor -32/3 \rfloor = -11$.

   (g) $q(12, 11)$
   Answer: $q(12, 11) = \lfloor 12/11 \rfloor = 1$.

   (h) $q(-12, 11)$
   Answer: $q(-12, 11) = \lfloor -12/11 \rfloor = -2$.

11. (Understand how to extend the quotient function to negative divisors.) Let $a \in \mathbb{Z}$ and $n \in \mathbb{Z}$, $n \neq 0$ be a non-zero integer The quotient function $q(a, n)$ can be given by

$$q(a, n) = \left\lfloor \frac{a}{n} \right\rfloor$$

Compute the following quotients.

(a) $q(7, -2)$
Answer: $q(7, -2) = \lfloor 7/-2 \rfloor = -4$.

(b) $q(-7, -2)$
Answer: $q(7, -2) = \lfloor -7/-2 \rfloor = -3$.

(c) $q(5, -6)$
Answer: $q(5, -6) = \lfloor 5/-6 \rfloor = -1$.

(d) $q(-5, -6)$
Answer: $q(-5, -6) = \lfloor -5/-6 \rfloor = 0$.

(e) $q(32, -3)$
Answer: $q(32, -3) = \lfloor 32/-3 \rfloor = -11$.

(f) $q(-32, -3)$
Answer: $q(-32, -3) = \lfloor -32/-3 \rfloor = 10$.

(g) $q(12, -11)$
Answer: $q(12, -11) = \lfloor 12/-11 \rfloor = 2$.

(h) $q(-12, -11)$
Answer: $q(12, -11) = \lfloor -12/-11 \rfloor = 1$.

12. (Understand and be able to compute the mod function.) The mod function is the remainder when a is divided by n. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, $n \neq 0$ be an integer and a non-zero natural number. The mod function a mod n is given by

$$a \bmod n = a - nq(a, n) = a - n \left\lfloor \frac{a}{n} \right\rfloor$$

Compute the following mods.

(a) 7 mod 2
Answer: $7 \bmod 2 = 7 - 2q(7, 2) = 7 - 2 \cdot 3 = 1$.

(b) -7 mod 2
Answer: $-7 \bmod 2 = -7 - 2q(-7, 2) = -7 - 2 \cdot (-4) = 1$.

(c) 5 mod 6
Answer: $5 \bmod 6 = 5 - 6q(5, 3) = 7 - 2 \cdot 3 = 1$.

(d) -5 mod 6
Answer: $-5 \bmod 6 = -5 - 6q(-5, 6) = 7 - 2 \cdot 3 = 1$.

(e) 32 mod 3
Answer: $32 \bmod 3 = 32 - 3q(32, 3) = 32 - 3 \cdot 10 = 2$.

(f) -32 mod 3
Answer: $-32 \bmod 3 = -32 - 3q(-32, 3) = -32 - 3 \cdot (1 - 1) = 1$.

(g) 12 mod 11)
Answer: $12 \bmod 11 = 12 - 11q(-32, 3) = 12 - 11 \cdot (11) = 1$.

(h) -12 mod 11)
Answer: $-12 \bmod 11 = -12 - 11q(-32, 3) = -12 - 3 \cdot (-11) = 1$.

13. (Understand how to extend the mod function to a negative modulus.) The mod function is the remainder when a is divided by n. Let $a \in \mathbb{Z}$ and $n \in \mathbb{Z}$, $n \neq 0$. The mod function a mod n is given by

$$a \bmod n = a - nq(a, n) = a - n \left\lfloor \frac{a}{n} \right\rfloor$$

Compute the following mods.

(a) $7 \bmod -2$
Answer: $7 \bmod 2 = 7 - 2q(7, 2) = 7 - 2 \cdot 3 = 1$.

(b) $-7 \bmod -2$
Answer: $-7 \bmod 2 = -7 - 2q(-7, 2) = -7 - 2 \cdot (-4) = 1$.

(c) $5 \bmod -3$
Answer: $5 \bmod 3 = 5 - 3q(5, 3) = 7 - 2 \cdot 3 = 1$.

(d) $-5 \bmod -3$
Answer: $-5 \bmod 3 = 7 - 2q(7, 2) = 7 - 2 \cdot 3 = 1$.

(e) $32 \bmod -3$
Answer: $32 \bmod 3 = 32 - 3q(32, 3) = 32 - 3 \cdot 10 = 2$.

(f) $-32 \bmod -3$
Answer: $-32 \bmod 3 = -32 - 3q(-32, 3) = -32 - 3 \cdot (-11) = 1$.

14. (Be able to extend the mod function to $n = 1$) How would you define $a \bmod 1$ for $a \in \mathbb{Z}$?
Answer: By the definition, $a \bmod 1 = a - 1 \lfloor a/1 \rfloor = 0$.

15. (Be able to extend the mod function to $n = 0$) How would you define $a \bmod 0$?
Answer: $a \bmod 0$ when $a - b$ is a multiple of 0. This only happens when $a = b$.

Problems on Permutations

Background

Permutations are one-to-one functions from a set onto itself.

1. (Know the basic property of a permutation.) `True` or `False`: A permutation of $\mathbb{X}$ is a one-to-one function from $\mathbb{X}$ onto $\mathbb{X}$.
Answer: This is `True`.

2. (Be able to identify permutations) Which of the following are permutations?

(a) `True` or `False`: $\langle c, a, b \rangle$ is a permutation of $\{a, b, c\}$.
Answer: This is `True`.

(b) `True` or `False`: $\langle a, a, b \rangle$ is a permutation of $\{a, b, c\}$.
Answer: This is `False`. A permutation of $\{a, b, c\}$ is one of 6 orderings of the letters a, b, and c.

3. (Be able to correctly count the number of permutations of n different things)

(a) `True` or `False`: There are 2^n permutations of an n element set.
Answer: This is `False`. There are $n!$ permutations of n elements.

(b) `True` or `False`: There are $n!$ permutations of an n element set.
Answer: This is `True`.

4. (Know that $n!$ counts the number of different permutations on an n element set.) How many permutations can be defined on the following sets? If there are fewer than 20, list each of them.

(a) $\emptyset$
Answer: There is $0! = 1$ permutation of the members of the empty set $\emptyset$. Intuitively, there is 1 way to arrange nothing at all. This is the "empty" function that accepts no input and produces no output.

(b) $\mathbb{B}$
Answer: There is $2! = 2$ permutation of the bits: $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$.

(c) $\{a, b, c\}$
Answer: There are $3 \cdot 2! = 6$ permutations of 3 different values:

$$\begin{array}{ll} \langle a, b, c\rangle & \langle a, c, b\rangle \\ \langle b, a, c\rangle & \langle b, c, a\rangle \\ \langle c, a, b\rangle & \langle c, b, a\rangle \end{array}$$

(d) $\mathbb{D}$
Answer: There are $10! = 3,628,800$ permutations of the 10 digits.

(e) $\mathbb{H}$?
Answer: There are $16! = 20,922,789,888,000 \approx 2.1 \times 10^{13}$ permutations of the 16 hexadecimal numerals.

5. (Understand the use of binomial coefficients and Stirling numbers of the first kind) Let $\mathbb{H} = \{0, 1, 2, \ldots, E, F\}$.

(a) In how many ways can you choose 5 elements from $\mathbb{H}$
Answer:

$$\binom{16}{5}$$

(b) In how many ways can you choose and permute 5 elements from $\mathbb{H}$
Answer:

$$\binom{16}{5} 5! = 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$$

(c) How many permutations on $\mathbb{H}$ have 16 cycles?
Answer:

$$\begin{bmatrix} 16 \\ 16 \end{bmatrix} = 1$$

(d) How many permutations on $\mathbb{H}$ have 1 cycle?
Answer:

$$\begin{bmatrix} 16 \\ 1 \end{bmatrix} = 15!$$

6. (Be able to write permutations using cyclic notation.) Use cyclic notation to describe the permutations of $\{0, 1, 2, 3\}$.

(a) $\langle 0, 1, 2, 3\rangle$
Answer:

$$[0][1][2][3]$$

(b) $\langle 1, 0, 3, 2 \rangle$
Answer:

$$[01][23]$$

(c) $\langle 3, 2, 1, 0 \rangle$
Answer:

$$[03]][21]$$

(d) $\langle 3, 1, 2, 0 \rangle$
Answer:

$$[03]][1][2]$$

7. (Be able to write permutations using cyclic notation.) Use cyclic notation to describe the permutation of of the octal numerals in the set $\mathbb{O} = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(a) $\langle 0, 2, 4, 6, 1, 3, 5, 7 \rangle$
Answer:

$$[0][142][356][7]$$

(b) $\langle 1, 2, 0, 7, 3, 4, 5, 6 \rangle$
Answer:

$$[021][34567]$$

8. (Be able to derive the permutation from its cyclic notation.) What is the permutation of the numerals in the set $\mathbb{O} = \{0, 1, 2, 3\}$. described by the cyclic notation.

(a) $[0][1][2][3]$
Answer:

$$\langle 0, 1, 2, 3 \rangle$$

(b) $[01][23]$
Answer:

$$\langle 1, 0, 3, 2 \rangle$$

(c) $[03]][21]$
Answer:

$$\langle 3, 2, 1, 0 \rangle$$

(d) $[03]][1][2]$
Answer:

$$\langle 3, 1, 2, 0 \rangle$$

9. (Be able to derive the permutation from its cyclic notation.) What is the permutation of the octal numerals in the set $\mathbb{O} = \{0, 1, 2, 3, 4, 5, 6, 7\}$. described by the cyclic notation.

   (a) $[0][142][356][7]$
   Answer:
   $$\langle 0, 2, 4, 6, 1, 3, 5, 7 \rangle$$

   (b) $[021][34567]$
   Answer:
   $$\langle 1, 2, 0, 7, 3, 4, 5, 6 \rangle$$

10. ⚠ (Be able use cyclic notation to write all permutations of small sets.)

   (a) What are the permutations of $\mathbb{B}$?
   Answer: There is one permutations with one cycle: $[0, 1]$. There is one permutation with two cycles: $[0][1]$.

   (b) What are the permutations of $\{0, 1, 2\}$?
   Answer: There are two permutations with one cycle: $[0, 1, 2]$ and $[0, 2, 1]$. There are three permutation with two cycles: $[0, 1][2]$, $[0, 2][1]$, and $[1, 2][0]$. There is one permutation with three cycles: $[0][1][2]$.

   (c) What are the permutations of $\{0, 1, 2, 3\}$?
   Answer: There are six permutations with one cycle: $[0, 1, 2, 3]$, $[0, 1, 3, 2]$, $[0, 3, 1, 2]$, $[0, 2, 1, 3]$, $[0, 2, 3, 1]$, and $[0, 3, 2, 1]$. There are eleven permutation with two cycles: $[0, 1, 2][3]$, $[0, 2, 1][3]$. $[0, 1][2, 3]$, $[0, 1, 3][2]$, $[0, 3, 1][2]$, $[0, 2][1, 3]$, $[0, 2, 3][1]$, $[0, 3, 2][1]$, $[1, 2][0, 3]$, $[1, 2, 3][0]$, $[1, 3, 2][0]$. There are six permutation with three cycles: $[0, 1][2][3]$, $[0, 2][1][3]$, $[1, 2][0][3]$, There is one permutation with four cycles: $[0][1][2][3]$.

11. ⚠ (Know that Stirling number of the first kind $\left[{n \atop m}\right]$ counts the number permutations of n objects with m cycles.) Using the results from problem 10 compute the following Stirling number of the second kind.

   (a) $\left[{2 \atop 1}\right]$
   Answer: There is $\left[{2 \atop 1}\right] = 1$ permutation of a 2 element set into 1 cycle.

   (b) $\left[{2 \atop 2}\right]$
   Answer: There is $\left[{2 \atop 2}\right] = 1$ permutation of a 2 element set into 2 cycles.

   (c) $\left[{3 \atop 1}\right]$
   Answer: There is $\left[{3 \atop 1}\right] = 2$ permutations of a 3 element set into 1 cycle.

   (d) $\left[{3 \atop 2}\right]$
   Answer: There are $\left[{3 \atop 2}\right] = 3$ permutations of a 3 element set into 2 cycles.

   (e) $\left[{3 \atop 3}\right]$
   Answer: There is $\left[{3 \atop 3}\right] = 1$ permutation of a 3 element set into 3 cycles.

   (f) $\left[{4 \atop 1}\right]$
   Answer: There is $\left[{4 \atop 1}\right] = 6$ permutations of a 4 element set into 1 cycle.

   (g) $\left[{4 \atop 2}\right]$
   Answer: There are $\left[{4 \atop 2}\right] = 11$ permutations of a 4 element set into 2 cycles.

(h) $\left[{4 \atop 3}\right]$
Answer: There are $\left[{4 \atop 3}\right] = 6$ permutations of a 4 element set into 3 cycles.

(i) $\left[{4 \atop 4}\right]$
Answer: There is $\left[{4 \atop 4}\right] = 1$ permutation of a 4 element set into 4 cycles.

12. (Know the recursion for Stirling numbers of the first kind.) Stirling's triangle of the first kind is

Stirling Numbers of the First Kind $\left[{n \atop m}\right]$

		Cycle m								
		0	1	2	3	4	5	6	7	8
	0	1	0	0	0	0	0	0	0	0
	1	0	1	0	0	0	0	0	0	0
	2	0	1	1	0	0	0	0	0	0
	3	0	2	3	1	0	0	0	0	0
n	4	0	6	11	6	1	0	0	0	0
	5	0	24	50	35	10	1	0	0	0
	6	0	120	274	225	85	15	1	0	0
	7	0	720	1764	1624	735	175	21	1	0
	8	0	5040	13068	13132	6769	1960	322	28	1

What is the recurrence equation and boundary conditions for the entry $\left[{n \atop m}\right]$ in row n, column m?
Answer: The entry $\left[{n \atop m}\right]$ can be computed by the recurrence equation

$$\begin{bmatrix} n \\ m \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$$

with boundary conditions $(\forall n \in \mathbb{N})(\left[{n \atop n}\right] = 1)$, $(\forall n \in \mathbb{N},\, n > 0)(\left[{n \atop 0}\right] = 0)$.

13. (Understand the recurrence for Stirling numbers of the first kind.) The set $\mathbb{A}_3 = \{0, 1, 2\}$ has 2 permutations with 1 cycle:

$$\langle 2, 0, 1\rangle = [012] \quad \text{and } \langle 1, 2, 0\rangle = [021]$$

and 3 permutation with 2 cycles:

$$\langle 1, 0, 2\rangle = [01][2],\ \langle 2, 1, 0\rangle = [02][1],\ \langle 0, 2, 1\rangle = [0][12]$$

Show how to use these permutations to construct all 2 cycle permutations of

$$\mathbb{A}_4 = \{0, 1, 2, 3\} = \mathbb{A}_3 \cup \{3\}$$

and by doing so demonstrate that Stirling's formula

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

holds in this instance.
Answer: Each 1 cycle permutation: $[012]$ and $[021]$ can be made into a 2 cycle permutation of $\{0, 1, 2, 3\}$ appending the cycle $[3]$ yielding

$$\langle 2, 0, 1, 3\rangle = [012][3] \quad \text{and } \langle 1, 2, 0, 3\rangle = [021][3]$$

Each of the 2 cycle permutations: $[01][2]$, $[02][1]$ and $[0][12]$ can be made into a two cycle permutation of $\mathbb{T}_4$ by inserting 3 in three possible ways.

$$\langle 1, 0, 3, 2\rangle = [01][23],\ \langle 3, 0, 2, 1\rangle = [013][2],\ \langle 1, 3, 2, 0\rangle = [031][2]$$
$$\langle 2, 3, 0, 1\rangle = [02][13],\ \langle 3, 1, 0, 2\rangle = [023][1],\ \langle 2, 1, 3, 0\rangle = [032][1]$$
$$\langle 3, 2, 1, 0\rangle = [03][12],\ \langle 0, 3, 1, 2\rangle = [0][123],\ \langle 0, 2, 3, 1\rangle = [0][132]$$

There are no other 2 cycle permutations of $\mathbb{A}_4$. Therefore

$$\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} + 3\begin{bmatrix}3\\2\end{bmatrix} = 2 + 3\cdot 3 = 11$$

holds in this instance.

14. (Be able to compute Stirling numbers of the first kind using the recurrence of these numbers.) Let $\mathbb{O}$ be the set of octal numerals, an 8-element set. How many permutations on $\mathbb{O}$ have 4 cycles? That is, what is the value of the Stirling number $\left[{8 \atop 4}\right]$ "8 cycle 4"? You may want to know row 7 of Stirling's triangle of the first kind is

	0	1	2	3	4	$\cdots$
7	0	720	1764	1624	735	$\cdots$

Answer: Using the recurrence

$$\begin{bmatrix}n\\m\end{bmatrix} = \begin{bmatrix}n-1\\m-1\end{bmatrix} + (n-1)\begin{bmatrix}n-1\\m\end{bmatrix}$$

for Stirling numbers of the first kind with $n = 8$ and $m = 4$ compute

$$\begin{bmatrix}8\\4\end{bmatrix} = \begin{bmatrix}7\\3\end{bmatrix} + 7\begin{bmatrix}7\\4\end{bmatrix} = 1624 + 7\cdot 735$$