In Class Problem Set #11

CSE 1400 and MTH 2051

Fall 2012

Instructions

1. Break any cliques that have been formed.
2. Follow the instructions given in previous problem sets.

Summations

1. What functions compute the following summations.

   Answer:

   (a) $\sum_{k=0}^{n-1} 1 = \underbrace{1+1+1+1+\cdots+1}_{n \text{ ones}}$

   Answer: The sum can be computed by the function $f(n) = n$, that is, the sum of n 1's is

   $$\sum_{k=0}^{n-1} 1 = n$$

   Note that

   $$\sum_{k=0}^{n-1} 1 = \left(\sum_{k=0}^{n-2} 1\right) + 1$$

   Let

   $$f(n) = \sum_{k=0}^{n-1} 1 = \underbrace{1+1+1+1+\cdots+1}_{n \text{ ones}}$$

   then $f(n)$ satisfies the recurrence equation

   $$f_n = f_{n-1} + 1$$

   the recurrence that enumerates the natural numbers.

   (b) $\sum_{k=0}^{n-1} k = 0+1+2+\cdots+(n-1)$

   Answer: The sum can be computed by the function $f(n) = n(n-1)/2$, that is, the sum $0+1+2+\cdot(n-1)$ is

   $$\sum_{k=0}^{n-1} k = \frac{n(n-1)}{2}$$

   Note that

   $$\sum_{k=0}^{n-1} k = \left(\sum_{k=0}^{n-2} k\right) + (n-1)$$

   Let

   $$f(n) = \sum_{k=0}^{n-1} k = 0+1+2+\cdots+(n-1)$$

   then $f(n)$ satisfies the recurrence equation

   $$f_n = f_{n-1} + (n-1)$$

   the recurrence that enumerates the triangular numbers.

   (c) $\sum_{k=0}^{n-1} \binom{k}{2} = \sum_{k=0}^{n-1} \frac{k(k-1)}{2} = \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \cdots + \binom{n-1}{2}$

   Answer: Note that

   $$\sum_{k=0}^{n-1} \binom{k}{2} = \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \cdots + \binom{n-1}{2}$$

   is the sum of entries in the column labeled 2 in Pascal's triangle. The first two terms are equal to zero $\binom{0}{2} = \binom{1}{2} = 0$. (This sum can be computed by the function

   Recall: A column of Pascal's triangle sums to the value in the next column, one row down.

   $$f(n) = \binom{n}{3} = n(n-1)(n-2)/6 = \binom{n}{3},$$

that is,

$$\sum_{k=0}^{n-1} \frac{k(k-1)}{2} = \frac{n(n-1)(n-2)}{6} = \binom{n}{3} = \frac{n^{\underline{3}}}{3!}$$

Here

$$n^{\underline{3}} = n(n-1)(n-2)$$

is a "falling factorial power" and pronounced "n falling 3."

(d) $\displaystyle\sum_{k=0}^{n-1} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \binom{2}{m} + \cdots + \binom{n-1}{m}$

Answer: Note this is the sum of entries in the column labeled m in Pascal's triangle. The sum can be computed by the function $f(n) = n(n-1)(n-2)\cdots(n-m+1)/m! = \binom{n}{m}$, that is,

$$\sum_{k=0}^{n-1} \binom{k}{m} = \binom{n}{m} = \frac{n^{\underline{m}}}{m!}$$

Here

$$n^{\underline{m}} = n(n-1)(n-2)\cdots(n-m+1)$$

is a "falling factorial power" and pronounced "n falling m."

(e) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

Answer: Note this is the sum of entries in the row labeled n in Pascal's triangle. The sum is the total number of subsets of an n element set, that is, 2^n. The sum $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ is

$$\sum_{k=0}^{n-1} \binom{n}{k} = 2^k$$

(f) $\displaystyle\sum_{k=0}^{n-1} 2^k = 1 + 2 + 4 + 8 + \cdots + 2^{n-1}$

Answer: The sum can be computed by the function $f(n) = 2^n - 1$, that is, the sum $2^0 + 2^1 + 2^2 + \cdot 2^{n-1}$ is

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$