In Class Problem Set #15

CSE 1400 and MTH 2051

Fall 2012

Relations

Definitions

A relation is a set of ordered pairs (x, y) where x is "related" to y. Let $\sim$ denote a relational symbol. Write $x \sim y$ to express the predicate x is related to y.

- A relation is <u>reflexive</u> if for all x, $x \sim x$.
- A relation is <u>symmetric</u> if for all x and y, $x \sim y$ implies $y \sim x$.
- A relation is <u>antisymmetric</u> if for all x and y, $x \sim y$ and $y \sim x$ implies $x = y$.
- A relation is <u>transitive</u> if for all x, y and z, $x \sim y$ and $y \sim z$ implies $x \sim z$.

- Reflexive:

$$(\forall x)(x \sim x)$$

- Symmetric:

$$(\forall x, y)((x \sim y) \to (y \sim x))$$

- Antisymmetric:

$$(\forall x, y)(((x \sim y) \wedge (y \sim x)) \to (x = y))$$

- Transitive:

$$(\forall x, y, z)(((x \sim y) \wedge (y \sim z)) \to (x \sim z))$$

Problems

1. Use Boolean algebra to show that $(x \sim y) \to (y \sim x)$ is equivalent to $(y \not\sim x) \to (x \not\sim y)$.

   Answer:

$(x \sim y)$	$(y \sim x)$	$(x \sim y) \to (y \sim x)$	$(y \not\sim x)$	$\to$	$(x \not\sim y)$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	1	0

   Note that $(x \sim y) \to (y \sim x)$ and $(y \not\sim x) \to (x \not\sim y)$ are two equivalent ways to state $\sim$ is a symmetric relation.

2. Use Boolean algebra to show that $((x \sim y) \wedge (y \sim x)) \to (x = y)$ is equivalent to $(x \neq y) \to ((x \not\sim y) \vee (y \not\sim x))$.

Answer:

$(x \sim y)$	$(y \sim x)$	$x = y$	$(x \sim y) \wedge (y \sim x)$	$\rightarrow$	$x = y$	$x \neq y$	$\rightarrow$	$(x \not\sim y) \vee (y \not\sim x)$
0	0	0	0	1	0	1	1	1
0	0	1	0	1	1	0	1	1
0	1	0	0	1	0	1	1	1
0	1	1	0	1	1	0	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	1	0	1	1
1	1	0	1	0	0	1	0	0
1	1	1	1	1	1	0	1	0

Note that $((x \sim y) \wedge (y \sim x)) \rightarrow (x = y)$ and $(x \neq y) \rightarrow ((x \not\sim y) \vee (y \not\sim x))$ are two equivalent ways to state $\sim$ is an antisymmetric relation.

3. Use Boolean algebra to show that $((x \sim y) \wedge (y \sim z)) \rightarrow (x \sim z)$ equivalent to $(x \not\sim z) \rightarrow ((x \not\sim y) \vee (y \not\sim z))$

Answer:

$(x \sim y)$	$(y \sim z)$	$(x \sim z)$	$(x \sim y) \wedge (y \sim z)$	$\rightarrow$	$(x \sim z)$	$(x \not\sim z)$	$\rightarrow$	$(x \not\sim y) \vee (y \not\sim z)$
0	0	0	0	1	0	1	1	1
0	0	1	0	1	1	0	1	1
0	1	0	0	1	0	1	1	1
0	1	1	0	1	1	0	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	1	0	1	1
1	1	0	1	0	0	1	0	0
1	1	1	1	1	1	0	1	0

Note that $((x \sim y) \wedge (y \sim z)) \rightarrow (x \sim z)$ and $(x \not\sim z) \rightarrow ((x \not\sim y) \vee (y \not\sim z))$ are two equivalent ways to state $\sim$ is a transitive relation.

4. For each relation (on the set $\mathbb{R}$ of real numbers), identify if it is reflexive, symmetric, antisymmetric, or transitive.

(a) Equality: $x = y$

Answer:

- $x = x$ is `True`, so equality is reflexive
- If $x = y$, then $y = x$ is `True`, so equality is symmetric
- If $x = y$ and $y = x$, then $x = y$ is `True`, so equality is antisymmetric
- If $x = y$ and $y = z$, then $x = z$ is `True`, so equality is transitive

(b) Not equal: $x \neq y$

Answer:

- $x \neq x$ is `False`, so not equal is not reflexive

- If $x \neq y$, then $y \neq x$ is `True`, so not equal is symmetric
- If $x \neq y$ and $y \neq x$, then $x = y$ is `False`, so equal is not antisymmetric
- If $x \neq y$ and $y \neq z$, then $x \neq z$ is `False`, so not equality is not transitive

(c) Less than: $x < y$

Answer:

- $x < x$ is `False`, so less than is not reflexive
- If $x < y$, then $y < x$ is `False`, so less than is not symmetric
- If ($x < y$ and $y < x$), then $x = y$ is `True`, so less than is antisymmetric (The conditional is `True` because the premise ($x < y$ and $y < x$) is `False`)
- If $x < y$ and $y < z$, then $x < z$ is `True`, so less than is transitive

(d) Greater than or equal: $x \geq y$

Answer:

- $x \geq x$ is `True`, so greater than or equal is reflexive
- If $x \geq y$, then $y \geq x$ is `False`, so greater than or equal is not symmetric
- If ($x \geq y$ and $y \geq x$), then $x = y$ is `True`, so greater than or equal is antisymmetric
- If $x \geq y$ and $y \geq z$, then $x \geq z$ is `True`, so greater than or equal is transitive

(e) Equal magnitude: $|x| = |y|$

Answer:

- $|x| = |x|$ is `True`, so equal magnitude is reflexive
- If $|x| = |y|$, then $|y| = |x|$ is `True`, so equal magnitude is symmetric
- If ($|x| = |y|$ and $|y| = |x|$), then $x = y$ is `False`, so equal magnitude is not antisymmetric
- If $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$ is `True`, so equal magnitude is transitive

(f) Approximately equal (within $\epsilon > 0$): $|x - y| \leq \epsilon$

Answer:

- $|x| = |x|$ is `True`, so equal magnitude is reflexive
- If $|x| = |y|$, then $|y| = |x|$ is `True`, so equal magnitude is symmetric
- If ($|x| = |y|$ and $|y| = |x|$), then $x = y$ is `False`, so equal magnitude is not antisymmetric

- If $|x| = |y|$ and $|y| = |z|$, then $|x| = |z|$ is `True`, so equal magnitude is transitive

5. For each relation (on the set of geometric figures), identify if it is reflexive, symmetric, antisymmetric, or transitive.

   (a) Parallel lines: $L_0 \| L_1$

   Answer:

   - $L_0 \| L_0$ is `True`, so parallel lines is reflexive
   - If $L_0 \| L_1$, then $L_1 \| L_0$ is `True`, so parallel lines is symmetric
   - If ($L_0 \| L_1$ and $L_1 \| L_0$), then $L_0 = L_1$ is `False`, so parallel lines is not antisymmetric
   - If $L_0 \| L_1$ and $L_1 \| L_2$, then $L_0 \| L_2$ is `True`, so parallel lines is transitive

   (b) Perpendicular lines: $L_0 \perp L_1$

   Answer:

   - $L_0 \perp L_0$ is `False`, so perpendicular lines is not reflexive
   - If $L_0 \perp L_1$, then $L_1 \perp L_0$ is `True`, so perpendicular lines is symmetric
   - If ($L_0 \perp L_1$ and $L_1 \perp L_0$), then $L_0 = L_1$ is `False`, so perpendicular lines is not antisymmetric
   - If $L_0 \perp L_1$ and $L_1 \perp L_2$, then $L_0 \perp L_2$ is `False`, so perpendicular lines is not transitive

   (c) Similar triangles: $T_0 \sim T_1$ (every angle in T_0 is equal to a corresponding angle in T_1)

   Answer:

   - $T_0 \sim T_0$ is `True`, so similar triangles is reflexive
   - If $T_0 \sim T_1$, then $T_1 \sim T_0$ is `True`, so similar triangles is symmetric
   - If ($T_0 \sim T_1$ and $T_1 \sim T_0$), then $T_0 = T_1$ is `False`, so similar triangles is not antisymmetric
   - If $T_0 \sim T_1$ and $T_1 \sim T_2$, then $T_0 \sim T_2$ is `True`, so similar triangles is transitive

   (d) Congruent triangles: $T_0 \cong T_1$ (every angle and side in T_0 is equal to a corresponding angle and side in T_1)

   Answer:

   - $T_0 \cong T_0$ is `True`, so congruent triangles is reflexive
   - If $T_0 \cong T_1$, then $T_1 \cong T_0$ is `True`, so congruent triangles is symmetric
   - If ($T_0 \cong T_1$ and $T_1 \cong T_0$), then $T_0 = T_1$ is `False`, so congruent triangles is not antisymmetric

- If $T_0 \cong T_1$ and $T_1 \cong T_2$, then $T_0 \cong T_2$ is `True`, so congruent triangles is transitive