

Sign the exam with your student number - not your name \_\_\_\_\_

Answer all five questions to the best of your ability.

1. (20 pts) Let  $p(x, y)$  denote the statement:  $y - x = y + x^2$ , where the universe for the variables  $x$  and  $y$  is the set of all integers. Determine the truth value of the following statements.

1.  $p(0, 0)$

2.  $\exists y p(1, y)$

3.  $\forall x \exists y p(x, y)$

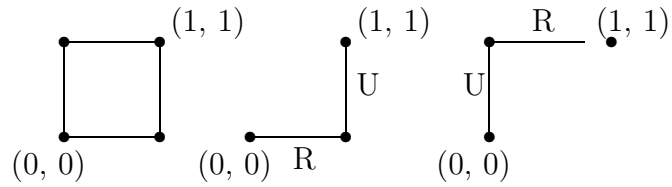
2. (20 pts) The *harmonic numbers*  $H_1, H_2, H_3, \dots$  are defined by

$$H_1 = 1, \quad H_2 = 1 + \frac{1}{2}, \quad H_3 = 1 + \frac{1}{2} + \frac{1}{3}, \quad \dots \quad H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

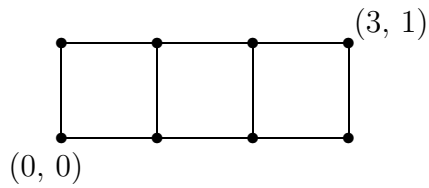
Use the principle of mathematical induction to prove that the sum of the first  $n$  harmonic numbers equals  $(n + 1)H_n - n$ , that is:

$$H_1 + H_2 + H_3 + \dots + H_n = \sum_{i=1}^n H_i = (n + 1)H_n - n.$$

3. (20 pts) Consider the problem of *counting* “Manhattan” paths in a grid. That is starting from  $(0, 0)$  with destination  $(n, m)$ ,  $n, m > 0$  you can move up (U) or right (R) to reach the destination. For examples, there are 2 paths from  $(0, 0)$  to  $(1, 1)$ : RU and UR



1. How many “Manhattan” paths are there from  $(0, 0)$  to  $(3, 1)$ ?



2. How many “Manhattan” paths are there from  $(0, 0)$  to  $(n, m)$ ?

4. (20 pts) Let  $T$  be a binary tree.

1. If  $T$  has height  $h$ , what is the smallest number of leaf nodes  $T$  can contain?

2. If  $T$  has height  $h$ , what is the largest number of leaf nodes  $T$  can contain?

3. If  $T$  has  $n$  leaf nodes, what is the smallest height  $T$  can have?

4. If  $T$  has  $n$  leaf nodes, is there a largest height  $T$  can have?

5. (20 pts) Some unpaved roads in Yoknapatawpha County must be paved so there is a paved path between every city in the county. The road construction company charges \$100,000 a mile. You serve on the County Council and must recommend which of the unpaved roads (shown below) to pave so that citizens can drive along some paved roads between towns. Use only the distances between towns (shown on the roads) in making your recommendation (do not use political favors, graft, or illicit methods in your decision making). Indicate which road to pave and the total cost to pave the roads you select?

