Discrete Mathematics **Comprehensive Examination, Fall 2003**

Sign the exam with your student number - not your name _____

Answer the following questions to the best of your ability.

1. (20 pts) For the universe of all people, consider the following statements:

   $P(x) = x$ is a professor
   $D(x) = x$ is dumb
   $V(x) = x$ is vain

   Express the following assumptions by appropriately using the symbols: $\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, and $\Rightarrow$

   1. A_1: Everyone is a professor or dumb.

   2. A_2: Everyone who is not a professor and dumb is vain.

   3. A_3: Some people are professors and vain.

   4. Using the assumptions, prove or disprove: All people who are not vain are professor.

2. (20 pts) Let $\Sigma = A, C, G, T$ be a four letter alphabet. A word w over Σ is a string of zero or more letters from Σ. The set of all words is denoted Σ^*.

   1. How many words are there of length 7?

   2. How many words of length 7 have at least one A?

   3. How many words of length 7 have exactly one A?

   4. How many words of length 7 have no letters that occur more than once?

3. (15 pts) Use the principle of mathematical induction to prove that the sum of cubes is a perfect square, that is, specifically:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

4. (10 pts) Use recursion to define the set E of all nonnegative even integers.

5. (20 pts) Let M be a deterministic finite automaton with alphabet Σ, start state s, and transition function $T : Q \times \Sigma^* \to Q$.

   Define a relation R on the strings over Σ by the statement

   "string $x \in \Sigma^*$ related to string $y \in \Sigma^*$ if and only if $T(s, x) = T(s, y)$"

   that is, xRy (string x is related to string y) when following their transitions from the start state s leads to the same exact state.

   1. Show this relation is *reflexive*.

   2. Show this relation is *symmetric*.

   3. Show this relation is *transitive*.

   4. Describe the equivalence classes induced by the relation R? In particular, describe the equivalence class in terms of states $q \in Q$ and words $x \in \Sigma^*$.

6. (15 pts) Consider the floor plan of a building diagrammed below.

   1. Develop a graph to model the rooms and doorway openings between them.

   2. Is there circuit through the floor plan that starts and ends outside, and passes through each doorway opening exactly once? "Yes" or "No" is not sufficient: Explain your answer.

   3. There is some path through that passes through each doorway opening exactly once. "Yes" or "No" is not sufficient: Explain your answer.