Discrete Mathematics **Comprehensive Examination, Fall 2004**

Sign the exam with your student number - not your name ____________________

Answer the following questions to the best of your ability.

1. (30 pts) Let $\mathbb{U} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set for this problem.

   1. What is the cardinality of $\mathbb{U}$?

   2. What is the cardinality of the power set of $\mathbb{U}$?

   3. Let $\mathbb{A} = \{0, 2, 4, 6, 8\}$. You can construct 4 subsets of the universal set $\mathbb{U}$ from $\mathbb{A}$, its complement $\overline{\mathbb{A}}$, union and intersection.

      $$\mathbb{A}, \quad \overline{\mathbb{A}}, \quad \emptyset = \mathbb{A} \cap \overline{\mathbb{A}}, \quad \mathbb{U} = \mathbb{A} \cup \overline{\mathbb{A}}$$

      1. Given 2 subsets $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{U}$ what is the most number of sets that can be formed using the sets, their complements, unions and intersections?

      2. What is the general formula, when given n subsets of a universal set (not necessarily the set $\mathbb{U}$ given above), for the most number of sets that can be formed using the sets, their complements, unions and intersections?

   4. Show that "subset" is a reflexive relation.

   5. Show that "subset" is an antisymmetric relation.

   6. Show that "subset" is an transitive relation.

   7. What name is applied to a relation that is reflexive, antisymmetric, and transitive?

2. (20 pts) Let $\mathbb{U} = \{1, 2, 3, \ldots, n\}$.

1. When $n = 1$, how many subsets of $\mathbb{U} = \{1\}$ contain no consecutive integers? Call this number S_1.

2. When $n = 2$, how many subsets of $\mathbb{U} = \{1, 2\}$ contain no consecutive integers? Call this number S_2.

3. When $n = 3$, how many subsets of $\mathbb{U} = \{1, 2, 3\}$ contain no consecutive integers? Call this number S_3.

4. Find a recurrence relation that counts S_n, the number of subsets of $\mathbb{U}$ that contain no consecutive integers, in terms of previous values of S_k, $k < n$?

5. What is the name of these numbers S_n, and what is a formula for them?

3. (20 pts) Prove that every graph has an even number of vertices with odd degree. As a hint, first determine a formula for the sum of degrees over all vertices of a graph in terms of the number of edges in the graph.

4. (20 pts) DNA is a chemical compound formed from fours bases designated as A, C, G and T.

   1. Triplets, three letter strings formed from the bases A, C, G and T play an important role in the function of DNA.

      1. If no restrictions are placed on the bases, how many triplets are there?

      2. Restricting the triplets so that any base occurs at most once in a triplet reduces the number of triplets to what value?

      3. Requiring some base to occur at exactly twice in a triplet produces how many triplets?

   2. An n-tuple is a string of length n formed from the bases A, C, G and T. Assume that $n \geq 3$.

      1. If no restrictions are placed on the bases, how many n-tuples are there?

      2. Under what conditions on n can you form an n-tuple so that no base occurs more than once?

      3. How many n-tuples are there where one and only one base occurs exactly twice?

5. (10 pts) Let $s = a_0a_1a_2 \cdots a_{n-2}a_{n-1}$ be a string of length n

1. When $n = 1$ so that $s = a_0$ in how many ways can you insert an open and close parenthesis () into s?

2. When $n = 2$ so that $s = a_0a_1$ in how many ways can you insert an open and close parenthesis () into s?

3. When $n = 3$ so that $s = a_0a_1a_2$ in how many ways can you insert an open and close parenthesis () into s?

4. Determine a formula for the numbers of ways to insert an open and close parenthesis () into s and prove your formula is correct using mathematical induction.

Total Points: 100