**Discrete Mathematics** **Comprehensive Examination, Fall 2005**

Sign the exam with your student number - not your name ______________________

Answer the following questions to the best of your ability.

1. (20 pts) Construct a truth table for the Boolean expression $(p \wedge (p \implies q)) \implies q$. Discuss the significance of the expression and its truth values.

   Answer:

   | $p$ | $q$ | $p \implies q$ | $p \wedge p \implies q$ | $(p \wedge (p \implies q)) \implies q$ |
   |---|---|---|---|---|
   | 0 | 0 | 1 | 0 | 1 |
   | 0 | 1 | 1 | 0 | 1 |
   | 1 | 0 | 0 | 0 | 1 |
   | 1 | 1 | 1 | 1 | 1 |

   The expression is a tautology: It is called "modus ponens," which a primary rule of inference used in direct proofs of statements $q$ from an assumption $p$.

2. (20 pts) Consider the "divides" relation on the set of natural numbers $\mathbf{N}$

$$a|b \Longleftrightarrow \exists c \in \mathbf{N}, b\ ac$$

1. Show that "divides" is a partial order on the set $\mathbf{N}$

   **Reflexive:** Divides is reflexive: $a|a$ since $a = 1 \times a$

   **Symmetric:** Divides is antisymmetric: if $a|b$ and $b|a$, then $b = c \times a$ and $a = d \times b$ for some natural numbers $c$ and $d$. Therefore

   $$b = c \times d \times b$$

   which implies $c = d = 1$ and $a = b$.

   **Transitive:** Divides is transitive: if $a|b$ and $b|c$, then $b = d \times a$ and $c = e \times b$. Therefore,

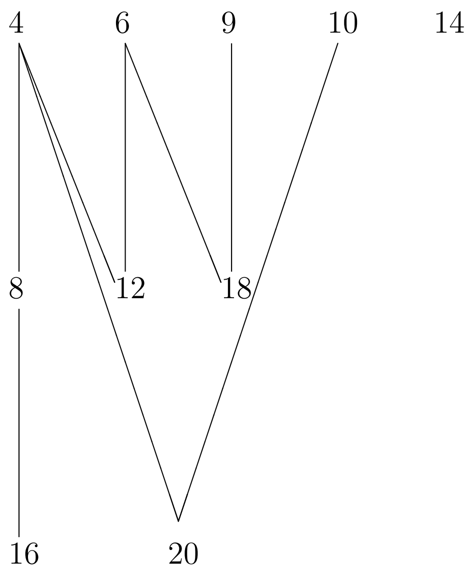
   $$c = e \times b = e \times d \times a$$

   and so $a|c$.

2. (20 pts) Draw a graph (a Hasse diagram) that shows how "divides" orders the numbers

   $$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20$$

   Answer:



3. (20 pts) Use (strong) mathematical induction to prove that the number of leaves in a binary tree is $(n+1)/2$ where $n$ is the number of nodes (assume integer arithmetic when $n$ is even).

**Basis:** A binary tree with $n = 1$ node has $(1+1)/2 = 1$ leaf.

**Strong Inductive Hypothesis:** Suppose that all binary trees with $n$ or fewer nodes have $(n+1)/2$ leaves, where $n \geq 1$.

**Inductive Step:** Let $T$ be a binary tree with $n+1$ nodes and let $L$ and $R$ be its left and right subtree. Both $L$ and $R$ have $n$ or fewer nodes, say they have $n_l$ and $n_r$ nodes where $n_l + n_r = n$. By the inductive hypothesis, $L$ has $(n_l+1)/2$ and $R$ has $(n_r+1)/2$ leaves. Therefore $T$ has

$$\frac{n_l+1}{2} + \frac{n_r+1}{2} = \frac{n_l+n_r+2}{2} = \frac{n+1}{2}$$

leaves.

4. (20 pts) Answer the following questions about binary strings.

    1. How many $n$-bit binary strings are there?
       Answer: $2^n$
    2. How many $n$-bit binary strings are there with exactly 3 "1" bits?
       Answer: $\binom{n}{3}$ for $n \geq 3$; 0 otherwise.
    3. How many $n$-bit binary strings are there with at most 3 "1" bits?
       Answer: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$ for $n \geq 3$; 2, 4 for $n = 1, 2$.
    4. How many $n$-bit binary strings have no consecutive "1" bits?

Answer: Consider how such strings can be formed:

$$\begin{array}{r|lllll} n=0 & \epsilon & & & & \\ n=1 & 0 & 1 & & & \\ n=2 & 01 & 00 & 10 & & \\ n=3 & 001 & 101 & 000 & 010 & 100 \end{array}$$

To get the next list of valid strings append 01 to all strings of length $n-2$: 0101, 0001, 1001 or append 0 to all strings of length $n-1$: 0010, 1010, 0000, 0100, 1000.

Therefore, the number of strings satisfying the condition is the Fibonacci number $F_{n+2}$, where $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, etc.

5. (20 pts) Consider the program

```
MM(int n) {
   if (n <= 2) { return 1; }
   else { return 2*MM(n/2) + 2; }
}
```

What is a formula for `MM(n)`?

Answer:

$$\begin{aligned}
\texttt{MM(n)} &= 2\texttt{MM(n/2)} + 2 \\
&= 2(2\texttt{MM(n/4)} + 2) + 2 \\
&= 2^2(2\texttt{MM(n/8)} + 2) + 2^2 + 2 \\
\vdots\ &\ \ \vdots \\
&= 2^k(2\texttt{MM(n/}2^{k+1}\texttt{)} + 2) + \sum_{i=1}^{k} 2^i \\
&= 2^k(2 + 2) + \sum_{i=1}^{k} 2^i \\
&= 2^{k+1} + \sum_{i=1}^{k+1} 2^i \\
&= 2^{k+1} + 2^{k+2} - 2
\end{aligned}$$

where $n/2^{k+1} = 2$ or $n = 2^{k+2}$, so that

$$\texttt{MM}(n) = 1.5n - 2$$