Discrete Mathematics **Comprehensive Examination, Spring 2006**

Sign the exam with your student number - not your name _____

Answer the following questions to the best of your ability.

1. (5^{pts}) What is a simple formula for the geometric sum $1 + 2 + 4 + 8 + \cdots + 2^n$?

2. (5^{pts}) What is the largest unsigned integer that can be represented with $n + 1$ bits?

3. (5^{pts}) Let $B(x_0, x_1, \ldots, x_n)$ be a Boolean expression on $n+1$ variables. In how many ways can "true" or "false" be assigned to the $n+1$ variables $x_0, x_1, \ldots, x_n$ if they are not all assigned the value "true"?

4. (5^{pts}) A complete binary tree is a binary tree where each internal node has 2 children and all leafs are at the same height. How many nodes are in a complete binary tree of height n?

5. (10^{pts}) Given that each of the following is true:

> "If T is a complete binary tree, then the number of nodes in T is a Mersenne number."
> "A tree T is a complete binary tree, only if no internal node of T has only one child."
> "The number of nodes in T is a Mersenne number."

Can we validly conclude:

> "No internal node of T has only one child."

If yes, then derive the conclusion using logical reasoning. Otherwise, explain why the conclusion cannot be drawn.

6. (10^{pts}) On the set of binary trees, define tree T to be "structurally similar" to tree S if both T and S are complete binary trees or neither T nor S is a complete binary tree. Is "structurally similar" an equivalence relation? Carefully explain your answer.

7. (10^{pts}) Show how to solve the recurrence equation

$$T_n = 2T_{n-1} + 1 \qquad \text{where} \quad T_0 = 1$$

8. (10^{pts}) Can a complete binary tree contain an Euler path? Carefully explain your answer.

9. (10^{pts}) Can a complete binary tree contain a Hamilton path? Carefully explain your answer.

10. (10^{pts}) A path from the root to a leaf in a complete binary tree can be described by a sequences of "L's" and "R's" indicate whether a left branch or right branch was taken. For complete binary tree of height n, how many paths are there with exactly k left branches for $0 \leq k \leq n$?

11. (10^{pts}) Give a recursive definition of a complete binary tree.

12. (10^{pts}) Use mathematical induction to prove that a complete binary tree of height n has $f(n)$ nodes, where $f(n)$ is the formula you gave in problem (4).