Discrete Mathematics **Comprehensive Examination, Fall 2009**

Sign the exam with your student number - not your name ______________________

Answer the following questions to the best of your ability.

1. (20^{pts}) Give a simple formulas that can be used to evaluate each of the following summations.

$$0 + 1 + 2 + 3 + 4 + \cdots + (n-1) = \sum_{k=0}^{n-1} k$$

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1} = \sum_{k=0}^{n-1} 2^k$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$$

$$\log(1) + \log(2) + \log(3) + \cdots + \log(n) = \sum_{k=1}^{n} \log(k)$$

2. (20^{pts}) Let $\mathbb{B} = \{0, 1\}$ be the set of bits, and let $\mathbb{B}^n$ be the Cartesian product of $\mathbb{B}$ with itself n times. A Boolean function b on n Boolean variables maps $\mathbb{B}^n$ to $\mathbb{B}$ ($b : \mathbb{B}^n \longrightarrow \mathbb{B}$).

   (a) (5) What is the cardinality of $\mathbb{B}$?

   (b) (5) What is the cardinality of $\mathbb{B}^n$?

   (c) (10) How many Boolean functions on n variables are there?

3. (10^{pts}) For $n \geq 0$, the triangular sequence is

$$0,\ 0,\ 1,\ 3,\ 6,\ 10, \ldots,\ n(n-1)/2,\ \ldots$$

Use mathematical induction to show that the partial sums of triangular sequence equals a tetrahedral number $n(n-1)(n-2)/6$. That is, prove by mathematical induction that

$$\sum_{k=0}^{n-1} \frac{1}{2} k(k-1) = \frac{1}{6} n(n-1)(n-2), \quad \text{for all } n \geq 0$$

4. (10^{pts}) Let $\mathbb{Q}$ be the set of rational numbers. Let a/b and c/d be rational numbers. Write

$$\frac{a}{b} \oplus \frac{c}{d} \quad \text{if } a + d = b + c$$

For example $1/2 \oplus 4/5$. Show that $\oplus$ is an equivalence relation on $\mathbb{Q}$.

5. (10^{pts}) Solve the recurrence equation

$$T_n = T_{n-1} + n \qquad \text{where} \quad T_0 = 0$$

6. (10^{pts}) Let $\mathbb{P}$ be a set of pigeons and let $\mathbb{H}$ be set of holes such that $|\mathbb{P}| > |\mathbb{H}|$.

   (a) Prove that if f is a function from $\mathbb{P}$ to $\mathbb{H}$ then f is not one-to-one.

   (b) Let p be a prime number and let $f(x) = x \pmod{p}$ be a hash function. Under what conditions will this hash function certainly produce a collision?

7. (10^{pts}) Give a recursive definition for binary trees.

8. (10^{pts}) Let T be a binary tree. Let $h(T)$ be the height of T and let $l(T)$ be the number of leaves in T.

   For binary trees T_1 and T_2 define

   $$T_1 \leq T_2 \quad \text{if } h(T_1) < h(T_2) \text{ or } h(T_1) = h(T_2) \text{ and } l(T_1) \leq l(T_2)$$

   Show that $\leq$ is a partial order on binary trees.