Discrete Mathematics Comprehensive Examination

Department of Computer Sciences, College of Engineering,
Florida Institute of Technology

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Sign the exam with your student number — not your name. Each of the five sections are equally weighted.

Relations

Consider the relation

$$R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

from the set $\{1, 2, 3, 4\}$ to the set $\{1, 2, 3, 4\}$.

1. Draw a directed graph that depicts relation R.

2. Represent relation R as an adjacency matrix M.

3. Is relation R reflexive? Is it symmetric? Is it antisymmetric?

4. For a reflexive relation, what property will its directed graph have? What property will its adjacency matrix have?

5. For a symmetric relation, what property will its directed graph have? What property will its adjacency matrix have?

6. For an antisymmetric relation, what property will its directed graph have? What property will its adjacency matrix have?

Summations

1. What is the formula for the sum of the first n natural numbers?

$$\sum_{k=0}^{n-1} k = 0 + 1 + 2 + \cdots + (n-1)$$

2. What is the formula for the sum of the first n powers of 2?

$$\sum_{k=0}^{n-1} 2^k = 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}$$

3. What is the formula for the sum of the binomial coefficients?

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}$$

4. What is the formula for the sum of the first n Fibonacci numbers?

$$\sum_{k=0}^{n-1} F_k = 0 + 1 + 1 + \cdots + F_{n-1}$$

Induction

Use mathematical induction to prove the sum

$$\sum_{k=0}^{n} k2^k = 0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n$$

is equal to the function $s(n) = (n-1)2^{n+1} + 2$.

Logic

Let the domain be the set $\mathbb{P}$ of all people. Let

- $\textsc{b}(x)$ be the predicate "x is a baby."
- $\textsc{l}(x)$ be the predicate "x is logical."
- $\textsc{c}(x)$ be the predicate "x is able to manage a crocodile."
- $\textsc{d}(x)$ be the predicate "x is despised."

Express each of these statements using quantifiers; logical operations; and $\textsc{b}(x)$, $\textsc{l}(x)$, $\textsc{c}(x)$, and $\textsc{d}(x)$.

1. Babies are illogical.

2. Nobody is despised who can manage a crocodile.

3. Illogical persons are despised.

4. Babies cannot manage a crocodile.

Does 4 logically follow from 1, 2, and 3? If so, explain why. If not, is there a correct conclusion?

Counting

1. How many permutations are there of n distinct things?

2. Cyclic notation can be used to represent permutations. For example,

Cyclic Notation	Permutation	Interpretation	Cycles
[123]	$3, 1, 2$	1 goes to 2, 2 goes to 3, 3 goes to 1	1
[132]	$2, 3, 1$	1 goes to 3, 2 goes to 1, 3 goes to 2	1
[1][23]	$1, 3, 2$	1 goes to 1, 2 goes to 3, 3 goes to 2	2
[2][13]	$3, 2, 1$	1 goes to 3, 2 goes to 2, 3 goes to 1	2
[3][12]	$2, 1, 3$	1 goes to 2, 2 goes to 1, 3 goes to 3	2
[1][2][3]	$1, 2, 3$	1 goes to 1, 2 goes to 2, 3 goes to 3	3

Show how to construct all eleven 2-cycle permutations of $\{1, 2, 3, 4\}$ from 1 and 2-cycle permutations above.

3. Let $\left[{n \atop k}\right]$ denote the number of k-cycle permutations of n distinct things, for instance $\left[{4 \atop 2}\right] = 11$. Generalize your discovery in problem 2 to develop a recurrence relation that computes $\left[{n \atop k}\right]$ in terms of $\left[{n-1 \atop k-1}\right]$ and $\left[{n-1 \atop k}\right]$.