Discrete Mathematics **Comprehensive Examination** **Fall 2013**

Sign the exam with your student number — Not your name____________________________

Answer the following questions to the best of your ability.

1. Functions

Let $\mathbb{X}$ and $\mathbb{Y}$ be sets with cardinalities n and m, respectively.

1. How many different functions $f : \mathbb{X} \to \mathbb{Y}$ are there?
2. How many of these functions are one-to-one?
3. Let the domain $\mathbb{X}$ be the set of all n-tuples $(b_0, b_1, \ldots, b_{n-1})$ of Boolean values.
   (a) What is the cardinality of $\mathbb{X}$ in this case?
   (b) Let the co-domain be $\mathbb{Y} = \{0, 0.5, 1\} = \{\texttt{False}, \texttt{Maybe}, \texttt{True}\}$. How many (quasi-Boolean) functions can be defined from $\mathbb{X}$ to $\mathbb{Y}$?
   (c) It is estimated that there are about 10^{80} hydrogen atoms in the observable universe. Approximately how large must n be for there to be more quasi-Boolean functions than hydrogen atoms?

2. *Combinatorics*

Let $\mathbb{E} = \{a, b, c, \ldots, z\}$ be the set of lowercase English letters, and let $\mathbb{E}^*$ be the set of all strings over $\mathbb{E}$. Given a file $\langle F \rangle$ that contains 700 strings from $\mathbb{E}^*$, separated by commas, are the following two statements `True` or `False`? You must explain your answer.

1. If all strings are one or two characters long there are duplicate strings in $\langle F \rangle$.
2. If all strings are three or fewer characters there are duplicate strings in $\langle F \rangle$.

3. *Recursion & Induction*

Consider the sequence

$$\langle F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \ldots \rangle = \langle 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \rangle$$

1. What is the recurrence equation that defines terms in the sequence?
2. Prove that

   $$F_{n+1} = F_1 F_{n+1} + F_0 F_n$$

   and

   $$F_{n+2} = F_2 F_{n+1} + F_1 F_n$$

3. Prove that

   $$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

   for all values of $k = 1, 2, 3, 4, \ldots$.

4. Relations

Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be be two points in $\mathbb{R} \times \mathbb{R}$. Say that p and q are *cross-sum* related if $a + d = b + c$. For instance, $(2, 5)$ is cross-sum related to $(4, 7)$ since $2 + 7 = 5 + 4$. (You can use the notation $(a, b) \oplus (c, d)$ to express that two points are cross-sum related.) Show that cross-sum is an equivalence relation.