Discrete Mathematics Comprehensive Examination Spring 2014

Sign the exam with your student number — Not your name______________________________

Answer the following questions to the best of your ability.

1. *Relations & Functions*

1. Let $p = (a, b)$ and $q = (c, d)$ be be two points in $\mathbb{R} \times \mathbb{R}$. Say, that p and q are *homogeneously* related if $ad = bc$. Use the notation $(a, b) \oplus (c, d)$ to express the homogeneous relation.
   (a) Is the relation a partial order?
   (b) Is the relation an equivalence?

2. Let $\mathbb{X}$ and $\mathbb{Y}$ be sets with cardinalities n and m, respectively.

   (a) How many different functions $f : \mathbb{X} \to \mathbb{Y}$ can be defined?

   (b) How many of these functions are one-to-one?

   (c) Let the domain $\mathbb{X}$ be the set of all n-tuples $(b_0, b_1, \ldots, b_{n-1})$ of quasi-Boolean values. That is, each b_k can be assigned a value from the set $\{$`False`, `Maybe`, `True`$\}$.

      i. What is the cardinality of $\mathbb{X}$ in this case?

      ii. Let the co-domain be the set of bits: $\mathbb{Y} = \mathbb{B} = \{0, 1\}$. How many (quasi-Boolean) functions can be defined from $\mathbb{X}$ to $\mathbb{Y}$?

      iii. How large would n need to be to have more functions than Internet Protocal version 6 address?

2. *Combinatorics*

Let $\mathbb{E} = \{a, b, c, \ldots, z\}$ be the set of lowercase English letters, and let $\mathbb{E}^*$ be the set of all strings over $\mathbb{E}$. Given a file $\langle F \rangle$ that contains 700 strings from $\mathbb{E}^*$, separated by commas, are the following two statements `True` or `False`? You must explain your answer.

1. If all strings are one or two characters long there must be duplicate strings in $\langle F \rangle$.
2. If all strings are three characters long there are no duplicate strings in $\langle F \rangle$.

3. *Recursion & Induction*

Consider the Lucas sequence

$$\vec{L} = \langle L_0, L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, \ldots \rangle = \langle 2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots \rangle$$

Let

$$\vec{F} = \langle F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, \ldots \rangle$$

be the Fibonacci sequence.

1. What recurrence equation and initial conditions define terms in the sequence $\vec{L}$?
2. Prove that

   $$L_{m+1} = L_{m+1}F_1 + L_m F_0$$

   and

   $$L_{m+2} = L_{m+1}F_2 + L_m F_1$$

3. Prove that

   $$L_{m+k} = L_{m+1}F_k + L_m F_{k-1}$$

   for all values of $k = 1, 2, 3, 4, \ldots$.

4. Proofs

Show that there are infinitely many prime numbers.