Nearly Optimal Sparse Fourier Transform

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Abstract

We consider the problem of computing the k-sparse approximation to the discrete Fourier transform of an n-dimensional signal. We show:

- An $O(k \log n)$ -time algorithm for the case where the input signal has at most k non-zero Fourier coefficients, and
- An $O(k \log n \log(n/k))$ -time algorithm for general input signals.

Both algorithms achieve $o(n \log n)$ time, and thus improve over the Fast Fourier Transform, for any k = o(n). Further, they are the first known algorithms that satisfy this property. Also, if one assumes that the Fast Fourier Transform is optimal, the algorithm for the exactly k-sparse case is optimal for any $k = n^{\Omega(1)}$.

We complement our algorithmic results by showing that any algorithm for computing the sparse Fourier transform of a general signal must use at least $\Omega(k \log(n/k)/\log \log n)$ signal samples, even if it is allowed to perform *adaptive* sampling.

1 Introduction

The discrete Fourier transform (DFT) is one of the most important and widely used computational tasks. Its applications are broad and include signal processing, communications, and audio/image/video compression. Hence, fast algorithms for DFT are highly valuable. Currently, the fastest such algorithm is the Fast Fourier Transform (FFT), which computes the DFT of an *n*-dimensional signal in $O(n \log n)$ time. The existence of DFT algorithms faster than FFT is one of the central questions in the theory of algorithms.

A general algorithm for computing the exact DFT must take time at least proportional to its output size, i.e., $\Omega(n)$. In many applications, however, most of the Fourier coefficients of a signal are small or equal to zero, i.e., the output of the DFT is (approximately) *sparse*. This is the case for video signals, where a typical 8x8 block in a video frame has on average 7 non-negligible frequency coefficients (i.e., 89% of the coefficients are negligible) [CGX96]. Images and audio data are equally sparse. This sparsity provides the rationale underlying compression schemes such as MPEG and JPEG. Other sparse signals appear in computational learning theory [KM91, LMN93], analysis of Boolean functions [KKL88, O'D08], multi-scale analysis [DRZ07], compressed sensing [Don06, CRT06], similarity search in databases [AFS93], spectrum sensing for wideband channels [LVS11], and datacenter monitoring [MNL10].

For sparse signals, the $\Omega(n)$ lower bound for the complexity of DFT no longer applies. If a signal has a small number k of non-zero Fourier coefficients – the *exactly k-sparse* case – the output of the Fourier transform can be represented succinctly using only k coefficients. Hence, for such signals, one may hope for a DFT algorithm whose runtime is sublinear in the signal size, n. Even for a general n-dimensional signal x – the general case – one can find an algorithm that computes the best k-sparse approximation of its Fourier transform, \hat{x} , in sublinear time. The goal of such an algorithm is to compute an approximation vector \hat{x}' that satisfies the following ℓ_2/ℓ_2 guarantee:

$$\|\widehat{x} - \widehat{x}'\|_{2} \le C \min_{k \text{-sparse } y} \|\widehat{x} - y\|_{2}, \tag{1}$$

where C is some approximation factor and the minimization is over k-sparse signals.

The past two decades have witnessed significant advances in sublinear sparse Fourier algorithms. The first such algorithm (for the Hadamard transform) appeared in [KM91] (building on [GL89]). Since then, several sublinear sparse Fourier algorithms for complex inputs were discovered [Man92, GGI⁺02, AGS03, GMS05, Iwe10, Aka10, HIKP12]. These algorithms provide¹ the guarantee in Equation (1).²

The main value of these algorithms is that they outperform FFT's runtime for sparse signals. For very sparse signals, the fastest algorithm is due to [GMS05] and has $O(k \log^c(n) \log(n/k))$ runtime, for some³ c > 2. This algorithm outperforms FFT for any k smaller than $\Theta(n/\log^a n)$ for some a > 1. For less sparse signals, the fastest algorithm is due to [HIKP12], and has $O(\sqrt{nk} \log^{3/2} n)$ runtime. This algorithm outperforms FFT for any k smaller than $\Theta(n/\log^3 n)$ for some a > 1.

Despite impressive progress on sparse DFT, the state of the art suffers from two main limitations:

- 1. None of the existing algorithms improves over FFT's runtime for the whole range of sparse signals, i.e., k = o(n).
- 2. Most of the aforementioned algorithms are quite complex, and suffer from large "big-Oh" constants (the algorithm of [HIKP12] is an exception, albeit with running time that is polynomial in *n*).

¹The algorithm of [Man92], as stated in the paper, addresses only the exactly k-sparse case. However, it can be extended to the general case using relatively standard techniques.

 $^{^{2}}$ All of the above algorithms, as well as the algorithms in this paper, need to make some assumption about the precision of the input; otherwise, the right-hand-side of the expression in Equation (1) contains an additional additive term. See Preliminaries for more details.

³The paper does not estimate the exact value of c. We estimate that $c \approx 3$.

Results. In this paper, we address these limitations by presenting two new algorithms for the sparse Fourier transform. Assume that the length n of the input signal is a power of 2. We show:

- An $O(k \log n)$ -time algorithm for the exactly k-sparse case, and
- An $O(k \log n \log(n/k))$ -time algorithm for the general case.

The key property of both algorithms is their ability to achieve $o(n \log n)$ time, and thus improve over the FFT, for any k = o(n). These algorithms are the first known algorithms that satisfy this property. Moreover, if one assume that FFT is optimal and hence DFT cannot be solved in less than $O(n \log n)$ time, the algorithm for the exactly k-sparse case is *optimal*⁴ as long as $k = n^{\Omega(1)}$. Under the same assumption, the result for the general case is at most one $\log \log n$ factor away from the optimal runtime for the case of "large" sparsity $k = n/\log^{O(1)} n$.

Furthermore, our algorithm for the exactly sparse case (depicted as Algorithm 3.1 on page 5) is quite simple and has low big-Oh constants. In particular, our preliminary implementation of a variant of this algorithm is faster than FFTW, a highly efficient implementation of the FFT, for $n = 2^{22}$ and $k \le 2^{17}$. In contrast, for the same signal size, prior algorithms were faster than FFTW only for $k \le 2000$ [HIKP12].⁵

We complement our algorithmic results by showing that any algorithm that works for the general case must use at least $\Omega(k \log(n/k)/\log \log n)$ samples from x. The lower bound uses techniques from [PW11], which shows an $\Omega(k \log(n/k))$ lower bound for the number of *arbitrary* linear measurements needed to compute the k-sparse approximation of an n-dimensional vector \hat{x} . In comparison to [PW11], our bound is slightly worse but it holds even for *adaptive* sampling, where the algorithm selects the samples based on the values of the previously sampled coordinates.⁶ Note that our algorithms are *non-adaptive*, and thus limited by the more stringent lower bound of [PW11].

The $\Omega(k \log(n/k)/\log \log n)$ lower bound for the sample complexity shows that the running time of our algorithm ($O(k \log n \log(n/k))$) is equal to the sample complexity of the problem times (roughly) $\log n$. One would speculate that this logarithmic discrepancy is due to the need of using FFT to process the samples. Although we do not have an evidence of the optimality of our general algorithm, the "sample complexity times $\log n$ " bound appears to be a natural barrier to further improvements.

Techniques – **overview.** We start with an overview of the techniques used in prior works. At a high level, sparse Fourier algorithms work by binning the Fourier coefficients into a small number of bins. Since the signal is sparse in the frequency domain, each bin is likely⁷ to have only one large coefficient, which can then be located (to find its position) and estimated (to find its value). The binning has to be done in sublinear time, and thus these algorithms bin the Fourier coefficients using an *n*-dimensional filter vector *G* that is concentrated both in time and frequency. That is, *G* is zero except at a small *number* of time coordinates, and its Fourier transform \hat{G} is negligible except at a small *fraction* (about 1/k) of the frequency coordinates, representing the filter's "pass" region. Each bin essentially receives only the frequencies in a narrow range corresponding to the pass region of the (shifted) filter \hat{G} , and the pass regions corresponding to different bins are disjoint. In this paper, we use filters introduced in [HIKP12]. Those filters (defined in more detail in Preliminaries) have the property that the value of \hat{G} is "large" over a constant fraction of the pass region,

⁴One also need to assume that k divides n. See section 5 for more details.

⁵Note that both numbers ($k \le 2^{17}$ and $k \le 2000$) are for the exactly k-sparse case. The algorithm in [HIKP12], however, can deal with the general case but the empirical runtimes are higher.

⁶Note that if we allow *arbitrary* adaptive linear measurements of a vector \hat{x} , then its k-sparse approximation can be computed using only $O(k \log \log(n/k))$ samples [IPW11]. Therefore, our lower bound holds only where the measurements, although adaptive, are limited to those induced by the Fourier matrix. This is the case when we want to compute a sparse approximation to \hat{x} from samples of x.

⁷One can randomize the positions of the frequencies by sampling the signal in time domain appropriately [GGI⁺02, GMS05]. See Preliminaries for the description.

referred to as the "super-pass" region. We say that a coefficient is "isolated" if it falls into a filter's superpass region and no other coefficient falls into filters pass region. Since the super-pass region of our filters is a constant fraction of the pass region, the probability of isolating a coefficient is constant.

To achieve the stated running times, we need a fast method for locating and estimating isolated coefficients. Further, our algorithm is iterative, so we also need a fast method for updating the signal so that identified coefficients are not considered in future iterations. Below, we describe these methods in more detail.

New techniques – **location and estimation.** Our location and estimation methods depends on whether we handle the exactly sparse case or the general case. In the exactly sparse case, we show how to estimate the position of an isolated Fourier coefficient using only two samples of the filtered signal. Specifically, we show that the phase difference between the two samples is linear in the index of the coefficient, and hence we can recover the index by estimating the phases. This approach is inspired by the frequency offset estimation in orthogonal frequency division multiplexing (OFDM), which is the modulation method used in modern wireless technologies (see [HT01], Chapter 2).

In order to design an algorithm⁸ for the general case, we employ a different approach. Specifically, we use variations of the filter \hat{G} to recover the individual bits of the index of an isolated coefficient. This approach has been employed in prior work. However, in those papers, the index was recovered bit by bit, and one needed $\Omega(\log \log n)$ samples per bit to recover *all* bits correctly with constant probability. In contrast, in this paper we recover the index one *block of bits* at a time, where each block consists of $O(\log \log n)$ bits. This approach is inspired by the fast sparse recovery algorithm of [GLPS10]. Applying this idea in our context, however, requires new techniques. The reason is that, unlike in [GLPS10], we do not have the freedom of using arbitrary "linear measurements" of the vector \hat{x} , and we can only use the measurements induced by the Fourier transform.⁹ As a result, the extension from "bit recovery" to "block recovery" is the most technically involved part of the algorithm. See Section 4.1 for further intuition.

New techniques – updating the signal. The aforementioned techniques recover the position and the value of any isolated coefficient. However, during each filtering step, each coefficient becomes isolated only with constant probability. Therefore, the filtering process needs to be repeated to ensure that each coefficient is correctly identified. In [HIKP12], the algorithm simply performs the filtering $O(\log n)$ times and uses the median estimator to identify each coefficient with high probability. This, however, would lead to a running time of $O(k \log^2 n)$ in the k-sparse case, since each filtering step takes $k \log n$ time.

One could reduce the filtering time by subtracting the identified coefficients from the signal. In this way, the number of non-zero coefficients would be reduced by a constant factor after each iteration, so the cost of the first iteration would dominate the total running time. Unfortunately, subtracting the recovered coefficients from the signal is a computationally costly operation, corresponding to a so-called *non-uniform* DFT (see [GST08] for details). Its cost would override any potential savings.

In this paper, we introduce a different approach: instead of subtracting the identified coefficients from the *signal*, we subtract them directly from the *bins* obtained by filtering the signal. The latter operation can be done in time linear in the number of subtracted coefficients, since each of them "falls" into only one bin. Hence, the computational costs of each iteration can be decomposed into two terms, corresponding to filtering the original signal and subtracting the coefficients. For the exactly sparse case these terms are as follows:

⁸We note that although the two-sample approach employed in our algorithm works only for the exactly k-sparse case, our preliminary experiments show that using more samples to estimate the phase works surprisingly well even for general signals.

⁹In particular, the method of [GLPS10] uses measurements corresponding to a random error correcting code.

- The cost of filtering the original signal is $O(B \log n)$, where B is the number of bins. B is set to O(k'), where k' is the number of yet-unidentified coefficients. Thus, initially B is equal to O(k), but its value decreases by a constant factor after each iteration.
- The cost of subtracting the identified coefficients from the bins is O(k).

Since the number of iterations is $O(\log k)$, and the cost of filtering is dominated by the first iteration, the total running time is $O(k \log n)$ for the exactly sparse case.

For the general case, the cost of each iterative step is multiplied by the number of filtering steps needed to compute the location of the coefficients, which is $O(\log(n/B))$. We achieve the stated running time by carefully decreasing the value of B as k' decreases.

2 Preliminaries

This section introduces the notation, assumptions, and definitions used in the rest of this paper.

Notation. For an input signal $x \in \mathbb{C}^n$, its Fourier spectrum is denoted by \hat{x} . For any complex number a, we use $\phi(a)$ to denote the *phase* of a. For any complex number a and a real positive number b, the expression $a \pm b$ denotes any complex number a' such that $|a - a'| \leq b$. We use [n] to denote the set $\{1 \dots n\}$.

Definitions. The paper uses two tools introduced in previous papers: (pseudorandom) spectrum permutation [GGI⁺02, GMS05, GST08] and flat filtering windows [HIKP12].

Definition 2.1. We define the permutation $P_{\sigma,a,b}$ to be

$$P_{\sigma,a,b}x)_i = x_{\sigma i+a}\omega^{-bi}$$

so $\widehat{P_{\sigma,a,b}x} = P_{\sigma^{-1},b,a}\hat{x}$. We also define $\pi_{\sigma,b}(i) = \sigma(i-b) \mod n$, so $\widehat{P_{\sigma,a,b}x}_{\pi_{\sigma,b}(i)} = \widehat{x}_i \omega^{-a\pi_{\sigma,b}(i)}$.

Definition 2.2. We say that $(G, \widehat{G'}) = (G_{B,\delta,\alpha}, \widehat{G'}_{B,\delta,\alpha}) \in \mathbb{R}^n$ is a flat window function with parameters $B, \delta, and \alpha \text{ if } |\operatorname{supp}(G)| = O(\frac{B}{\alpha} \log(1/\delta)) and \widehat{G'} \text{ satisfies}$

- $\widehat{G'}_i = 1 \text{ for } |i| \le (1 \alpha)n/(2B)$
- $\widehat{G'}_i = 0$ for $|i| \ge n/(2B)$
- $\widehat{G'}_i \in [0,1]$ for all i
- $\left\|\widehat{G'} \widehat{G}\right\|_{\infty} < \delta.$

The above notion corresponds to the $(1/(2B), (1-\alpha)/(2B), \delta, O(B/\alpha \log(1/\delta)))$ -flat window function in [HIKP12]. In Section 7 we give efficient constructions of such window functions, where G can be computed in $O(\frac{B}{\alpha} \log(1/\delta))$ time and for each i, $\widehat{G'}_i$ can be computed in $O(\log(1/\delta))$ time. Of course, for $i \notin [(1-\alpha)n/(2B), n/(2B)], \widehat{G'}_i \in \{0, 1\}$ can be computed in O(1) time.

We note that the simplest way of using the window functions is to precompute them once and for all (i.e., during a preprocessing stage dependent only on n and k, not x) and then lookup their values as needed, in constant time per value. However, the algorithms presented in this paper use the quick evaluation subroutines described in Section 7. Although the resulting algorithms are a little more complex, in this way we avoid the need for any preprocessing.

We use the following lemma about $P_{\sigma,a,b}$ from [HIKP12]:

Lemma 2.3 (Lemma 3.6 of [HIKP12]). If $j \neq 0$, n is a power of two, and σ is a uniformly random odd number in [n], then $\Pr[\sigma j \in [-C, C] \pmod{n}] \leq 4C/n$.

Assumptions. Through the paper, we assume that n, the dimension of all vectors, is an integer power of 2. We also make the following assumptions about the precision of the vectors \hat{x} :

- For the exactly k-sparse case, we assume that $\hat{x}_i \in \{-L, \ldots, L\}$ for some precision parameter L. To simplify the bounds, we assume that $L = n^{O(1)}$; otherwise the log n term in the running time bound is replaced by log L.
- For the general case, we assume that $\|\hat{x}\|_2 \le n^{O(1)} \cdot \min_{k \text{-sparse } y} \|\hat{x} y\|_2$. Without this assumption, we add $\delta \|\hat{x}\|_2$ to the right hand side of Equation (1) and replace $\log n$ by $\log(n/\delta)$ in the running time.

3 Algorithm for the exactly sparse case

Recall that we assume $\hat{x}_i \in \{-L \dots L\}$, where $L \leq n^c$ for some constant c > 0. We choose $\delta = 1/(16n^2L)$. The algorithm (NOISELESSSPARSEFFT) is described as Algorithm 3.1.

We analyze the algorithm "bottom-up", starting from the lower-level procedures.

Analysis of NOISELESSSPARSEFFTINNER. For any execution of NOISELESSSPARSEFFTINNER, define $S = \operatorname{supp}(\hat{x} - \hat{z})$. Recall that $\pi_{\sigma,b}(i) = \sigma(i - b) \mod n$. Define $h_{\sigma,b}(i) = \operatorname{round}(\pi_{\sigma,b}(i)B/n)$ and $o_{\sigma,b}(i) = \pi_{\sigma,b}(i) - h_{\sigma,b}(i)n/B$. Note that therefore $|o_{\sigma,b}(i)| \le n/(2B)$. We will refer to $h_{\sigma,b}(i)$ as the "bin" that the frequency *i* is mapped into, and $o_{\sigma,b}(i)$ as the "offset". For any $i \in S$ define two types of events associated with *i* and *S* and defined over the probability space induced by σ :

- "Collision" event $E_{coll}(i)$: holds iff $h_{\sigma,b}(i) \in h_{\sigma,b}(S \{i\})$, and
- "Large offset" event $E_{off}(i)$: holds iff $|o_{\sigma,b}(i)| \ge (1-\alpha)n/(2B)$.

Claim 3.1. For any $i \in S$, the event $E_{coll}(i)$ holds with probability at most 4|S|/B.

Proof. Consider distinct $i, j \in S$. By Lemma 2.3,

$$\Pr[|\pi_{\sigma,b}(i) - \pi_{\sigma,b}(j) \mod n| < n/B] \le \Pr[\sigma(i-j) \mod n \in [-n/B, n/B]] \le 4/B.$$

Hence $\Pr[h_{\sigma,b}(i) = h_{\sigma,b}(j)] < 4/B$, so $\Pr[E_{coll}(i)] \le 4|S|/B$.

Claim 3.2. For any $i \in S$, the event $E_{off}(i)$ holds with probability at most α .

Proof. Note that $o_{\sigma,b}(i) \equiv \pi_{\sigma,b}(i) \pmod{n/B}$. For any odd σ and $l \in [n/B]$, we have that $\Pr_b[\sigma(i-b) \equiv l \pmod{n/B}] = B/n$. The claim follows.

Lemma 3.3. The output \hat{u} of HASHTOBINS has

$$\widehat{u}_{j} = \sum_{h_{\sigma,b}(i)=j} \widehat{(x-z)_{j}} \widehat{(G_{B,\delta,\alpha})}_{-o_{\sigma,b}(i)} \omega^{-a\pi_{\sigma,b}(i)} \pm \delta(\|x\|_{1} + 2\|\widehat{z}\|_{1})$$

Let $\zeta = |\{i \in \operatorname{supp}(\widehat{z}) \mid E_{off}(i)\}|$. The running time of HASHTOBINS is $O(\frac{B}{\alpha}\log(1/\delta) + |\operatorname{supp}(\widehat{z})| + \zeta \log(1/\delta))$.

Proof. Define $G = G_{B,\delta,\alpha}$ and $G' = G_{B,\delta,\alpha}$. We have

$$\begin{split} \widehat{y} &= G \cdot \widehat{P_{\sigma,a,b}}(x) = \widehat{G} * \widehat{P_{\sigma,a,b}}(x) \\ \widehat{y'} &= \widehat{G} * \widehat{P_{\sigma,a,b}}(x-z) + (\widehat{G} - \widehat{G'}) * \widehat{P_{\sigma,a,b}}z \\ &= \widehat{G'} * \widehat{P_{\sigma,a,b}}(x-z) \pm \delta(\|x\|_1 + 2 \|z\|_1) \end{split}$$

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procedure HASHTOBINS $(x, \hat{z}, P_{\sigma,a,b}, B, \delta, \alpha)$ Compute $\widehat{y}_{jn/B}$ for $j \in [B]$, where $y = G_{B,\alpha,\delta} \cdot (P_{\sigma,a,b}(x))$ Compute $\widehat{y'}_{in/B} = \widehat{y}_{in/B} - (\widehat{G'_{B,\alpha,\delta}} * \widehat{P_{\sigma,a,b}z})_{in/B}$ for $j \in [B]$ **return** \widehat{u} given by $\widehat{u}_j = \widehat{y'}_{in/B}$. end procedure **procedure** NOISELESSSPARSEFFTINNER (x, k', \hat{z}) Let $B = k'/\beta$. Choose σ uniformly at random from the set of odd numbers in [n]. Choose *b* uniformly at random from [n]. $\widehat{u} \leftarrow \text{HASHTOBINS}(x, \widehat{z}, P_{\sigma,0,b}, B, \delta, \alpha).$ $\widehat{u}' \leftarrow \text{HASHTOBINS}(x, \widehat{z}, P_{\sigma,1,b}, B, \delta, \alpha).$ $\widehat{w} \leftarrow 0.$ Compute $J = \{j : |\hat{u}_j| > 1/2\}.$ for $j \in J$ do $a \leftarrow \widehat{u}_j / \widehat{u}'_j$. $i \leftarrow \pi_{\sigma, b}^{-1}(\operatorname{round}(\phi(a)n/(2\pi))).$ $v \leftarrow \operatorname{round}(\widehat{u}_i).$ $\widehat{w}_i \leftarrow v.$ end for return \widehat{w} end procedure **procedure** NOISELESSSPARSEFFT(x, k) $\widehat{z} \gets 0$ for $t \in 0, 1, ..., \log k$ do $k_t = k/2^t$. $\hat{z} \leftarrow \hat{z} + \text{NOISELESSSPARSEFFTINNER}(x, k_t, \hat{z}).$ for $i \in \operatorname{supp}(\widehat{z})$ do if $|z_i| \ge L$ then $z_i = 0$ end if end for end for return \hat{z} end procedure

Algorithm 3.1: Exact *k*-sparse recovery

Therefore

$$\begin{split} \widehat{u}_{j} &= \widehat{y'}_{jn/B} = \sum_{|l| < n/(2B)} \widehat{G}_{-l} (P_{\sigma, \overline{a, b}}(\overline{x} - z))_{jn/B + l} \pm \delta(||x||_{1} + 2 ||z||_{1}) \\ &= \sum_{|\pi_{\sigma, b}(i) - jn/B| < n/(2B)} \widehat{G}_{jn/B - \pi_{\sigma, b}(i)} (P_{\sigma, \overline{a, b}}(\overline{x} - z))_{\pi_{\sigma, b}(i)} \pm \delta(||x||_{1} + 2 ||z||_{1}) \\ &= \sum_{h_{\sigma, b}(i) = j} \widehat{G}_{-o_{\sigma, b}(i)} (\overline{x - z})_{i} \omega^{-a\pi_{\sigma, b}(i)} \pm \delta(||x||_{1} + 2 ||z||_{1}) \end{split}$$

We can compute HASHTOBINS via the following method:

- 1. Compute y with $|\operatorname{supp}(y)| = O(\frac{B}{\alpha}\log(1/\delta))$ in $O(\frac{B}{\alpha}\log(1/\delta))$ time. 2. Compute $v \in \mathbb{C}^B$ given by $v_i = \sum_j y_{i+jB}$.
- 3. As long as B divides n, by Claim 3.7 of [HIKP12] we have $\hat{y}_{jn/B} = \hat{v}_j$ for all j. Hence we can compute it with a *B*-dimensional FFT in $O(B \log B)$ time.
- 4. For each coordinate $i \in \operatorname{supp}(\widehat{z})$, decrease $\widehat{y_{h_{\sigma,b}(i)n/B}}$ by $\widehat{G'}_{o_{\sigma,b}(i)}\widehat{z}_i\omega^{-a\pi_{\sigma,b}(i)}$. This takes $O(|\operatorname{supp}(\widehat{z})| + \widehat{C'}_{o_{\sigma,b}(i)}\widehat{z}_i\omega^{-a\pi_{\sigma,b}(i)})$. $\zeta \log(1/\delta)$) time, since computing $\widehat{G'}_{o_{\sigma,b}(i)}$ takes $O(\log(1/\delta))$ time if $E_{off}(i)$ holds and O(1) otherwise.

Lemma 3.4. Consider any $i \in S$ such that neither $E_{coll}(i)$ nor $E_{off}(i)$ holds. Let $j = h_{\sigma,b}(i)$. Then

$$round(\phi(\hat{u}'_j/\hat{u}_j))n/(2\pi)) = \pi_{\sigma,b}(i),$$
$$round(\hat{u}_j) = \hat{x}_i - \hat{z}_i,$$

and $j \in J$.

Proof. We know that $||x||_1 \le nL$ and $||z||_1 \le nL$. Then by Lemma 3.3 and $E_{coll}(i)$ not holding,

$$\widehat{u}_j = (\widehat{x-z})_i \widehat{G}_{-o_{\sigma,b}(i)} \pm 3\delta nL.$$

Because $E_{off}(i)$ does not hold, $\hat{G}_{-o_{\sigma,b}(i)} = 1 \pm \delta$, so

$$\widehat{u}_j = (\widehat{x-z})_i \pm 3\delta nL \pm 2\delta L = (\widehat{x-z})_i \pm 4\delta nL.$$
⁽²⁾

Similarly,

$$\widehat{u}_j' = (\widehat{x-z})_i \omega^{-\pi_{\sigma,b}(i)} \pm 4\delta nL$$

Then because $4\delta nL < 1 \leq \left| \widehat{(x-z)_i} \right|$,

$$\phi(\widehat{u}_j) = 0 \pm \sin^{-1}(4\delta nL) = 0 \pm 8\delta nL$$

and $\phi(\widehat{u}'_i) = -\pi_{\sigma,b}(i) \pm 8\delta nL$. Thus $\phi(\widehat{u}_j/\widehat{u}'_i) = \pi_{\sigma,b}(i) \pm 16\delta nL = \pi_{\sigma,b}(i) \pm 1/n$. Therefore

round
$$(\phi(\widehat{u}'_j/\widehat{u}_j)n/(2\pi)) = \pi_{\sigma,b}(i).$$

Also, by Equation (2), round $(\hat{u}_i) = \hat{x}_i - \hat{z}_i$. Finally, $|\text{round}(\hat{u}_i)| = |\hat{x}_i - \hat{z}_i| \ge 1$, so $|\hat{u}_i| \ge 1/2$. Thus $j \in J$.

Claims 3.1 and 3.2 and Lemma 3.4 together guarantee that for each $i \in S$ the probability that P does not contain the pair $(i, (\hat{x} - \hat{z})_i)$ is at most $4|S|/B + \alpha$. We complement this observation with the following claim.

Claim 3.5. For any $j \in J$ we have $j \in h_{\sigma,b}(S)$. Therefore, $|J| = |P| \leq |S|$.

Proof. Consider any $j \notin h_{\sigma,b}(S)$. From the analysis in the proof of Lemma 3.4 it follows that $|\hat{u}_j| \leq 1$ $4\delta nL < 1/2.$

Lemma 3.6. Consider an execution of NOISELESSSPARSEFFTINNER, and let $S = \text{supp}(\hat{x} - \hat{z})$. If $|S| \leq k'$, then

$$E[\|\widehat{x} - \widehat{z} - \widehat{w}\|_0] \le 8(\beta + \alpha)|S|.$$

Proof. Let *e* denote the number of coordinates $i \in S$ for which either $E_{coll}(i)$ or $E_{off}(i)$ holds. Each such coordinate might not appear in *P* with the correct value, leading to an incorrect value of \hat{w}_i . In fact, it might result in an arbitrary pair (i', v') being added to *P*, which in turn could lead to an incorrect value of $\hat{w}_{i'}$. By Claim 3.5 these are the only ways that \hat{w} can be assigned an incorrect value. Thus we have

$$\|\widehat{x} - \widehat{z} - \widehat{w}\|_0 \le 2e$$

Since $E[e] \leq (4|S|/B + \alpha)|S| \leq (4\beta + \alpha)|S|$, the lemma follows.

Analysis of NOISELESSSPARSEFFT Consider the *t*th iteration of the procedure, and define $S_t = \operatorname{supp}(\hat{x} - \hat{z})$ where \hat{z} denotes the value of the variable at the beginning of loop. Note that $|S_0| = |\operatorname{supp}(\hat{x})| \le k$.

We also define an indicator variable I_t which is equal to 0 iff $|S_t|/|S_{t-1}| \le 1/8$. If $I_t = 1$ we say the the *t*th iteration was not *successful*. Let $\gamma = 8 \cdot 8(\beta + \alpha)$. From Lemma 3.6 it follows that $\Pr[I_t = 1 \mid |S_{t-1}| \le k/2^{t-1}] \le \gamma$. From Claim 3.5 it follows that even if the *t*th iteration is not successful, then $|S_t|/|S_{t-1}| \le 2$.

For any $t \ge 1$, define an event E(t) that occurs iff $\sum_{i=1}^{t} I_i \ge t/2$. Observe that if none of the events $E(1) \dots E(t)$ holds then $|S_t| \le k/2^t$.

Lemma 3.7. Let $E = E(1) \cup \ldots \cup E(\lambda)$ for $\lambda = 1 + \log k$. Assume that $(2e\gamma)^{1/2} < 1/4$. Then $\Pr[E] \le 1/3$. *Proof.* Let $t' = \lceil t/2 \rceil$. We have

$$\Pr[E(t)] \le \binom{t}{t'} \gamma^{t'} \le (te/t')^{t'} \gamma^{t'} \le (2e\gamma)^{t/2}$$

Therefore

$$\Pr[E] \le \sum_{t} \Pr[E(t)] \le \frac{(2e\gamma)^{1/2}}{1 - (2e\gamma)^{1/2}} \le 1/4 \cdot 4/3 = 1/3$$

Theorem 3.8. The algorithm NOISELESSSPARSEFFT runs in expected $O(k \log n)$ time and returns the correct vector \hat{x} with probability at least 2/3.

Proof. The correctness follows from Lemma 3.7. The running time is dominated by $O(\log k)$ executions of HASHTOBINS. Since

$$\mathbb{E}[|\{i \in \operatorname{supp}(z) \mid E_{off}(i)\}|] = \alpha |\operatorname{supp}(z)|,$$

the expected running time of each execution of HASHTOBINS is $O(\frac{B}{\alpha}\log n + k + \alpha k\log(1/\delta)) = O(\frac{B}{\alpha}\log n + k + \alpha k\log n)$. Setting $\alpha = \Theta(2^{-i/2})$ and $\beta = \Theta(1)$, the expected running time in round *i* is $O(2^{-i/2}k\log n + k + 2^{-i/2}k\log n)$. Therefore the total expected running time is $O(k\log n)$.

4 Algorithm for the general case

This section shows how to achieve Equation (1) for $C = 1 + \epsilon$. Pseudocode is in Algorithm 4.1 and 4.2.

4.1 Intuition

Let S denote the "heavy" $O(k/\epsilon)$ coordinates of \hat{x} . The overarching algorithm SPARSEFFT works by first finding a set L containing most of S, then estimating \hat{x}_L to get \hat{z} . It then repeats on $\widehat{x-z}$. We will show that each "heavy" coordinate has a large constant probability of both being in L and being estimated well. As a result, $\widehat{x-z}$ is probably k/4-sparse, so we can run the next iteration with $k \to k/4$. The later iterations will then run faster, so the total running time is dominated by the time in the first iteration.

Location As in the noiseless case, to locate the heavy coordinates we consider the bins computed by HASHTOBINS with $P_{\sigma,a,b}$. We have that each heavy coordinate *i* is probably alone in its bin, and would like to find its location $\tau = \pi_{\sigma,b}(i)$. In the noiseless case, we showed that the difference in phase in the bin using $P_{\sigma,0,b}$ and using $P_{\sigma,1,b}$ is $2\pi\frac{\tau}{n}$ plus a negligible $O(\delta)$ term. With noise this may not be true; however, we can say that the difference in phase between using $P_{\sigma,a,b}$ and $P_{\sigma,a+\beta,b}$, as a distribution over uniformly random *a*, is $2\pi\frac{\beta\tau}{n} + \nu$ with (for example) $\mathbb{E}[\nu^2] = 1/100$ (with all operations on phases modulo 2π). So our task is to find τ within a region *Q* of size n/k using $O(\log(n/k))$ "measurements" of this form.

One method for doing so would be to simply do measurements with random $\beta \in [n]$. Then each measurement lies within $\pi/4$ of $2\pi \frac{\beta \tau}{n}$ with at least $1 - \frac{\mathbb{E}[\nu^2]}{\pi^2/16} > 3/4$ probability. On the other hand, for $j \neq \tau$, $2\pi \frac{\beta \tau}{n} - 2\pi \frac{\beta j}{n}$ is roughly uniformly distributed around the circle. As a result, each measurement is probably more than $\pi/4$ away from $2\pi \frac{\beta j}{n}$. Hence $O(\log(n/k))$ repetitions suffice to distinguish among the n/k possibilities for τ . However, while the number of measurements is small, it is not clear how to decode in polylog rather than $\Theta(n/k)$ time.

To solve this, we instead do a *t*-ary search on the location for $t = O(\log n)$. At each of $O(\log_t(n/k))$ levels, we split our current candidate region Q into t consecutive subregions Q_1, \ldots, Q_t , each of size w. Now, rather than choosing $\beta \in [n]$, we choose $\beta \in [\frac{n}{16w}, \frac{n}{8w}]$. As a result, $\{2\pi \frac{\beta j}{n} \mid j \in Q_q\}$ all lie within a region of size $\pi/4$. On the other hand, if $|j - \tau| > 16w$, then $2\pi \frac{\beta \tau}{n} - 2\pi \frac{\beta j}{n}$ will still be roughly uniformly distributed about the circle. As a result, we can check a single candidate element e_q from each region: if e_q is in the same region as τ , each measurement usually agrees in phase; but if e_q is more than 16 regions away, each measurement usually disagrees in phase. Hence with $O(\log t)$ measurements, we can locate τ to within O(1) regions with failure probability $1/t^2$. The decoding time is $O(t \log t)$.

This primitive LOCATEINNER lets us narrow down the candidate region for τ to a subregion that is a $t' = \Omega(t)$ factor smaller. By repeating $\log_{t'}(n/k)$ times, we can find τ precisely. The number of measurements is then $O(\log t \log_t(n/k)) = O(\log(n/k))$ and the decoding time is $O(t \log t \log_t(n/k)) = O(\log(n/k) \log n)$. Furthermore, the "measurements" (which are actually calls to HASHTOBINS) are nonadaptive, so we can perform them in parallel for all $O(k/\epsilon)$ bins, with $O(\log(1/\delta)) = O(\log n)$ average time per bins per measurement.

Estimation By contrast, ESTIMATEVALUES is quite straightforward. Each measurement using $P_{\sigma,a,b}$ gives an estimate of each \hat{x}_i that is "good" with constant probability. However, we actually need each \hat{x}_i to be "good" with $1 - O(\epsilon)$ probability, since the number of candidates $|L| \approx k/\epsilon$. Therefore we repeat $O(\log \frac{1}{\epsilon})$ times and taking the median for each coordinate.

procedure SPARSEFFT (x, k, ϵ) $\widehat{z}^{(1)} \leftarrow 0$ for $r \in [R]$ do Choose B_r, k_r, α_r as in Theorem 4.9. $L_r \leftarrow \text{LocateSignal}(x, \hat{z}^{(r)}, B_r)$ $\hat{z}^{(r+1)} \leftarrow \hat{z}^{(r)} + \text{ESTIMATEVALUES}(x, \hat{z}^{(r)}, k_r, L_r, B_r).$ end for return $\widehat{z}^{(R+1)}$ end procedure **procedure** ESTIMATEVALUES (x, \hat{z}, k', L, B) for $r \in [R_{est}]$ do Choose $a_r, b_r \in [n]$ uniformly at random. Choose σ_r uniformly at random from the set of odd numbers in [n]. $\widehat{u}^{(r)} \leftarrow \text{HASHTOBINS}(x, \widehat{z}, P_{\sigma, a_r, b}, B, \delta).$ end for $\widehat{w} \leftarrow 0$ for $i \in L$ do $\widehat{w}_i \leftarrow \operatorname{median}_r \widehat{u}_{h_{\sigma,b}(i)}^{(r)} \omega^{a_r i}.$ end for $J \leftarrow \arg \max_{|J|=k'} \|\widehat{w}_J\|_2.$ return \widehat{w}_J end procedure

Algorithm 4.1: k-sparse recovery for general signals, part 1/2

4.2 Formal definitions

As in the noiseless case, we define $\pi_{\sigma,b}(i) = \sigma(i-b) \mod n$, $h_{\sigma,b}(i) = \operatorname{round}(\pi_{\sigma,b}(i)B/n)$ and $o_{\sigma,b}(i) = \pi_{\sigma,b}(i) - h_{\sigma,b}(i)n/B$. We say $h_{\sigma,b}(i)$ is the "bin" that frequency i is mapped into, and o_{σ} as the "offset". We define $h_{\sigma,b}^{-1}(j) = \{i \in [n] \mid h_{\sigma,b}(i) = j\}$.

Define

$$\operatorname{Err}(x,k) = \min_{k \text{-sparse } y} \left\| x - y \right\|_2.$$

In each iteration of SPARSEFFT, define $\widehat{x}' = \widehat{x} - \widehat{z}$, and let

$$\rho^{2} = \operatorname{Err}^{2}(\widehat{x'}, k) + \delta^{2} n^{3} (||x'||_{2}^{2} + ||x||_{2}^{2})$$
$$\mu^{2} = \epsilon \rho^{2} / k$$
$$S = \{i \in [n] \mid |\widehat{x'}_{i}|^{2} \ge \mu^{2}\}$$

Then $|S| \leq (1 + 1/\epsilon)k = O(k/\epsilon)$ and $\|\hat{x}' - \hat{x}'_S\|_2^2 \leq (1 + \epsilon)\rho^2$. We will show that each $i \in S$ is found by LOCATESIGNAL with probability $1 - O(\alpha)$, when $B = \Omega(\frac{k}{\alpha\epsilon})$.

For any $i \in S$ define three types of events associated with i and S and defined over the probability space induced by σ and a:

- "Collision" event $E_{coll}(i)$: holds iff $h_{\sigma,b}(i) \in h_{\sigma,b}(S \{i\})$;
- "Large offset" event $E_{off}(i)$: holds iff $|o_{\sigma}(i)| \ge (1-\alpha)n/(2B)$; and
- "Large noise" event $E_{noise}(i)$: holds iff $\left\| \widehat{x'}_{h_{\sigma,b}^{-1}(h_{\sigma,b}(i))\setminus S} \right\|_{2}^{2} \ge \rho^{2}/(\alpha B)$.

procedure LOCATESIGNAL(x, \hat{z}, B) Choose uniformly at random $b \in [n]$ and σ relatively prime to n. Initialize $l_i^{(1)} = (i-1)n/B$ for $i \in [B]$. Let $w_0 = n/B$, $t' = \log n$, t = 3t', $D_{max} = \log_{t'}(w_0 + 1)$. for $D \in [D_{max}]$ do $l^{(D+1)} \leftarrow \text{LOCATEINNER}(x, \hat{z}, B, \delta, \alpha, \sigma, \beta, l^{(D)}, w_0/(t')^{D-1}, t, R_{loc})$ end for $L \leftarrow \{\pi_{\sigma,b}^{-1}(l_j^{(D_{max}+1)}) \mid j \in [B]\}$ return L end procedure $\triangleright \delta, \alpha$ parameters for G, G' $\triangleright (l_1, l_1 + w), \dots, (l_B, l_B + w)$ the plausible regions. $\triangleright B \approx k/\epsilon$ the number of bins $\triangleright t \approx \log n$ the number of regions to split into. $\triangleright R_{loc} \approx \log t = \log \log n$ the number of rounds to run \triangleright Running time: $R_{loc}B\log(1/\delta) + R_{loc}Bt + R_{loc}|\operatorname{supp}(\widehat{z})|$ **procedure** LOCATEINNER($x, \hat{z}, B, \delta, \alpha, \sigma, b, l, w, t, R_{loc}$) Let $s = \Theta(\alpha^{1/3})$. Let $v_{j,q} = 0$ for $(j,q) \in [B] \times [t]$. for $r \in [R_{loc}]$ do Choose $a \in [n]$ uniformly at random. Choose $\beta \in \{\frac{snt}{4w}, \dots, \frac{snt}{2w}\}$ uniformly at random. $\widehat{u} \leftarrow \text{HASHTOBINS}(x, \widehat{z}, P_{\sigma, a, b}, B, \delta, \alpha).$ $\widehat{u}' \leftarrow \text{HASHTOBINS}(x, \widehat{z}, P_{\sigma, a+\beta, b}, B, \delta, \alpha).$ for $j \in [B]$ do $c_j \leftarrow \phi(\widehat{u}_j / \widehat{u}'_j)$ for $q \in [t]$ do $m_{j,q} \leftarrow l_j + \frac{q-1/2}{t} w$ $\theta_{j,q} \leftarrow \frac{2\pi\beta m_{j,q}}{n} \mod 2\pi$ if $\min(|\theta_{j,q} - c_j|, 2\pi - |\theta_{j,q} - c_j|) < s\pi$ then $v_{j,q} \leftarrow v_{j,q} + 1$ end if end for end for end for for $j \in [B]$ do $Q^* \leftarrow \{q : v_{j,q} > R_{loc}/2\}$ if $Q^* \neq \emptyset$ then $l'_j \leftarrow \min_{q \in Q^*} l_j + \frac{q-1}{t} w$ else $l'_j \leftarrow \perp$ end if end for return l' end procedure

Algorithm 4.2: *k*-sparse recovery for general signals, part 2/2

By Claims 3.1 and 3.2, $\Pr[E_{coll}(i)] \leq 2|S|/B = O(\alpha)$ and $\Pr[E_{off}(i)] \leq 2\alpha$ for any $i \in S$.

Claim 4.1. For any $i \in S$, $\Pr[E_{noise}(i)] \leq 8\alpha$.

Proof. For each $j \neq i$, $\Pr[h_{\sigma,b}(j) = h_{\sigma,b}(i)] \leq \Pr[|\sigma j - \sigma i| < n/B] \leq 4/B$ by Lemma 3.6 of [HIKP12]. Then $\mathbb{E}[\|\widehat{x'}_{j,-1}(j,-j)\|^2] \leq 4 \|\widehat{x'}_{j,-1}(j,-j)\|^2 / B \leq 4(1+\epsilon)\rho^2/B$

$$\mathbb{E}\left[\left\|\widehat{x'}_{h_{\sigma,b}^{-1}(h_{\sigma,b}(i))\setminus S}\right\|_{2}^{2}\right] \leq 4\left\|\widehat{x'}_{[n]\setminus S}\right\|_{2}^{2}/B \leq 4(1+\epsilon)\rho^{2}/B$$

The result follows by Chebyshev's inequality.

We will show that if $E_{coll}(i)$, $E_{off}(i)$, and $E_{noise}(i)$ all hold then SPARSEFFTINNER recovers \hat{x}'_i with constant probability.

Lemma 4.2. Let $a \in [n]$ uniformly at random and the other parameters be arbitrary in

$$\widehat{u} = \text{HashToBins}(x, \widehat{z}, P_{\sigma, a, b}, B, \delta, \alpha)_j$$

Then for any $i \in [n]$ with $j = h_{\sigma,b}(i)$ and not $E_{off}(i)$,

$$\mathbb{E}[\left|\widehat{u}_{j} - (\widehat{x-z})_{i}\right|^{2}] \leq 2(1+\delta)^{2} \left\|\widehat{(x-z)}_{h_{\sigma,b}^{-1}(j)\setminus\{i\}}\right\|_{2}^{2} + O(n\delta^{2})(\|x\|_{2}^{2} + \|\widehat{x-z}\|_{2}^{2})$$

Proof. Let $G = G_{B,\delta,\alpha}$. Let $T = h_{\sigma,b}^{-1}(j) \setminus \{i\}$. By Lemma 3.3,

$$\begin{aligned} \widehat{u}_{j} - \widehat{(x-z)}_{i} &= \sum_{i' \in T} \widehat{G}_{-o_{\sigma}(i)} \widehat{(x-z)}_{i'} \omega^{-a\pi_{\sigma,b}(i')} \pm O(\sqrt{n}\delta) (\|x\|_{2} + \left\|\widehat{x-z}\right\|_{2}) \\ \left|\widehat{u}_{j} - \widehat{(x-z)}_{i}\right| &\leq (1+\delta) \left|\sum_{i' \in T} \widehat{(x-z)}_{i'} \omega^{-a\pi_{\sigma,b}(i')}\right| + O(\sqrt{n}\delta) (\|x\|_{2} + \left\|\widehat{x-z}\right\|_{2}) \\ \left|\widehat{u}_{j} - \widehat{(x-z)}_{i}\right|^{2} &\leq 2(1+\delta)^{2} \left|\sum_{i' \in T} \widehat{(x-z)}_{i'} \omega^{-a\pi_{\sigma,b}(i')}\right|^{2} + O(n\delta^{2}) (\|x\|_{2} + \left\|\widehat{x-z}\right\|_{2})^{2} \\ \mathbb{E}\left[\left|\widehat{u}_{j} - \widehat{(x-z)}_{i}\right|^{2}\right] &\leq 2(1+\delta)^{2} \left\|\widehat{(x-z)}_{T}\right\|_{2}^{2} + O(n\delta^{2}) (\|x\|_{2}^{2} + \left\|\widehat{x-z}\right\|_{2}^{2}) \end{aligned}$$

where the last inequality is Parseval's theorem.

4.3 **Properties of LOCATESIGNAL**

Lemma 4.3. Let $T \subset [m]$ consist of t consecutive integers, and suppose $\sigma \in T$ uniformly at random. Then for any $i \in [n]$ and set $S \subset [n]$ of l consecutive integers,

$$\Pr[\sigma i \mod n \in S] \le \lceil im/n \rceil (1 + \lfloor l/i \rfloor)/t \le \frac{1}{t} + \frac{im}{nt} + \frac{lm}{nt} + \frac{l}{it}$$

Proof. Note that any interval of length l can cover at most $1 + \lfloor l/i \rfloor$ elements of any arithmetic sequence of common difference i. Then $\{\sigma i \mid \sigma \in T\} \subset [im]$ is such a sequence, and there are at most $\lceil im/n \rceil$ intervals an + S overlapping this sequence. Hence at most $\lceil im/n \rceil$ $(1 + \lfloor l/i \rfloor)$ of the $\sigma \in [m]$ have $\sigma i \mod n \in S$. Hence

$$\Pr[\sigma i \mod n \in S] \le \lceil im/n \rceil (1 + \lfloor l/i \rfloor)/t.$$

Lemma 4.4. Suppose none of $E_{coll}(i), E_{off}(i)$, and $E_{noise}(i)$ hold, and let $j = h_{\sigma,b}(i)$. Consider any run of LOCATEINNER with $\pi_{\sigma,b}(i) \in [l_j, l_j + w]$. Then $\pi_{\sigma,b}(i) \in [l'_j, l'_j + 3w/t]$ with probability at least $1 - tf^{\Omega(R_{loc})}$, as long as

$$B = \frac{Ck}{\alpha f \epsilon}$$

for C larger than some fixed constant.

Proof. Let $\tau = \pi_{\sigma,b}(i)$. Let $g = \Theta(f^{1/3})$, and $C' = \frac{B\alpha\epsilon}{k} = \Theta(1/g^3)$.

To get the result, we divide $[l_j, l_j + w]$ into t "regions", $Q_q = [l_j + \frac{q-1}{t}w, l_j + \frac{q}{t}w]$ for $q \in [t]$. We will first show that in each round r, c_j is close to $2\pi\beta\tau/n$ with large constant probability. This will imply that Q_q gets a "vote," meaning $v_{j,q}$ increases, with large constant probability for the q' with $\tau \in Q_{q'}$. It will also imply that $v_{j,q}$ increases with only a small constant probability when $|q - q'| \ge 2$. Then R_{loc} rounds will suffice to separate the two with "high" probability, allowing the recovery of q' to within 2, or the recovery of τ to within 3 regions or the recovery of τ within 3w/t.

Define $T = h_{\sigma,b}^{-1}(h_{\sigma,b}(i)) \setminus \{i\}$, so $\|\widehat{x'}_T\|_2^2 \leq \frac{\rho^2}{\alpha B}$. In any round r, define $\widehat{u} = \widehat{u}^{(r)}$ and $a = a_r$. We have by Lemma 4.2 that

$$\mathbb{E}[\left|\hat{u}_{j} - \omega^{-a\tau} \hat{x'}_{i}\right|^{2}] \leq 2(1+\delta)^{2} \left\|\hat{x'}_{T}\right\|_{2}^{2} + O(n\delta^{2})(\left\|x\right\|_{2}^{2} + \left\|\hat{x'}\right\|_{2}^{2})$$

$$< 3\frac{\rho^{2}}{\alpha B} \leq \frac{3k}{B\alpha\epsilon} |\hat{x'}_{i}|^{2}$$

$$= \frac{3}{C'} |\hat{x'}_{i}|^{2}.$$

Thus with probability 1 - p, we have

$$\left| \widehat{u}_j - \omega^{-a\tau} \widehat{x'}_i \right| \le \sqrt{\frac{3}{C'p}} \left| \widehat{x'}_i \right|$$
$$d(\phi(\widehat{u}_j), \phi(\widehat{x'}_i) - \frac{2\pi a\tau}{n}) \le \sin^{-1}(\sqrt{\frac{3}{C'p}})$$

where $d(x,y) = \min_{\gamma \in \mathbb{Z}} |x-y+2\pi\gamma|$ is the "circular distance" between x and y. The analogous fact holds for $\phi(\hat{u'}_j)$ relative to $\phi(\hat{x'}_i) - \frac{2\pi(a+\beta)\tau}{n}$. Therefore

$$d(\phi(\widehat{u}_j/\widehat{u'}_j), \frac{2\pi\beta\tau}{n})$$

$$=d(\phi(\widehat{u}_j) - \phi(\widehat{u'}_j), (\phi(\widehat{x'}_i) - \frac{2\pi a\tau}{n}) - (\phi(\widehat{x'}_i) - \frac{2\pi(a+\beta)\tau}{n}))$$

$$\leq d(\phi(\widehat{u}_j), \phi(\widehat{x'}_i) - \frac{2\pi a\tau}{n}) + d(\phi(\widehat{u'}_j), \phi(\widehat{x'}_i) - \frac{2\pi(a+\beta)\tau}{n})$$

$$< 2\sin^{-1}(\sqrt{\frac{3}{C'p}})$$

by the triangle inequality. Thus for any $s = \Theta(g)$ and $p = \Theta(g)$, we can set $C' = \frac{3}{p \sin^2(s\pi/4)} = O(1/g^3)$ so that

$$d(c_j, \frac{2\pi\beta\tau}{n}) < s\pi/2 \tag{3}$$

with probability at least 1 - 2p.

Equation (3) shows that c_j is a good estimate for *i* with good probability. We will now show that this means the appropriate "region" $Q_{q'}$ gets a "vote" with "large" probability.

For the q' with $\tau \in [l_j + \frac{q'-1}{t}w, l_j + \frac{q'}{t}w]$, we have that $m_{j,q'} = l_j + \frac{q'-1/2}{t}w$ satisfies

$$\left|\tau - m_{j,q'}\right| \le \frac{w}{2t}$$

and hence by Equation 3 and the triangle inequality,

$$d(c_j, \theta_{j,q'}) \le d(\frac{2\pi\beta\tau}{n}, c_j) + d(\frac{2\pi\beta\tau}{n}, \frac{2\pi\beta m_{j,q'}}{n})$$

$$< \frac{s\pi}{2} + \frac{2\pi\beta w}{2tn}$$

$$\le \frac{s\pi}{2} + \frac{s\pi}{2}$$

$$= s\pi$$

Thus, $v_{j,q'}$ will increase in each round with probability at least 1 - 2p.

Now, consider q with |q - q'| > 2. Then $|\tau - m_{j,q}| > \frac{(2 \cdot 2 + 1)w}{2t}$, and (from the definition of β) we have

$$\beta \left| \tau - m_{j,q} \right| > \frac{2(2+1)sn}{8} = 3sn/4.$$
(4)

We now consider two cases. First, assume that $|\tau - m_{j,q}| \leq \frac{w}{st}$. In this case, from the definition of β it follows that

$$\beta \left| \tau - m_{j,q} \right| \le n/2$$

Together with Equation (4) this implies

$$\Pr[\beta(\tau - m_{j,q}) \mod n \in [-3sn/4, 3sn/4]] = 0$$

On the other hand, assume that $|\tau - m_{j,q}| > \frac{w}{st}$. In this case, we use Lemma 4.3 with parameters l = 3sn/2, $m = \frac{snt}{2w}$, $t = \frac{snt}{4w}$, $i = (\tau - m_{j,q})$ and n to conclude that

$$\Pr[\beta(\tau - m_{j,q}) \mod n \in [-3sn/4, 3sn/4]] \le \frac{4w}{snt} + 2\frac{|\tau - m_{j,q}|}{n} + 3s + \frac{3sn}{2}\frac{st}{w}\frac{4w}{snt}$$
$$\le \frac{4w}{snt} + \frac{w}{n} + 9s$$
$$< \frac{5}{sB} + 9s < 10s$$

where we used that $|i| \le w/2 \le n/(2B)$, the assumption $\frac{w}{st} < |i|$, $t \ge 1$, s < 1, and that $s^2 > 5/B$ (because $s = \Omega(g)$ and $B = \omega(1/g^3)$). Thus in any case, with probability at least 1 - 10s we have

$$d(0, \frac{2\pi\beta(m_{j,q}-\tau)}{n}) > \frac{3}{2}s\pi$$

for any q with |q - q'| > 2. Therefore we have

$$d(c_j, \theta_{j,q}) \ge d(0, \frac{2\pi\beta(m_{j,q} - \tau)}{n}) - d(c_j, \frac{2\pi\beta\tau}{n}) > s\pi$$

with probability at least 1 - 10s - 2p, and $v_{j,q}$ is not incremented.

To summarize: in each round, $v_{j,q'}$ is incremented with probability at least 1-2p and $v_{j,q}$ is incremented with probability at most 10s + 2p for |q - q'| > 2. The probabilities corresponding to different rounds are independent.

Set s = f'/20 and p = f'/4. Then $v_{j,q'}$ is incremented with probability at least 1 - f' and $v_{j,q}$ is incremented with probability less than f'. Then after R_{loc} rounds, by the Chernoff bound, for |q - q'| > 2

$$\Pr[v_{j,q} > R_{loc}/2] \le \binom{R_{loc}}{R_{loc}/2} g^{R_{loc}/2} \le (4g)^{R_{loc}/2} = f^{\Omega(R_{loc})}$$

for $g = f^{1/3}/4$. Similarly,

$$\Pr[v_{j,q'} < R_{loc}/2] \le f^{\Omega(R_{loc})}.$$

Hence with probability at least $1 - t f^{\Omega(R_{loc})}$ we have $q' \in Q^*$ and $|q - q'| \leq 2$ for all $q \in Q^*$. But then $\tau - l'_i \in [0, 3w/t]$ as desired.

Because $\mathbb{E}[|\{i \in \operatorname{supp}(\widehat{z}) \mid E_{off}(i)\}|] = \alpha |\operatorname{supp}(\widehat{z})|$, the expected running time is $O(R_{loc}Bt + R_{loc}\frac{B}{\alpha}\log(1/\delta) + R_{loc}|\operatorname{supp}(\widehat{z})|$ $(1 + \alpha \log(1/\delta)))$.

Lemma 4.5. Suppose $B = \frac{Ck}{\alpha^2 \epsilon}$ for C larger than some fixed constant. Then for any $i \in S$, the procedure LOCATESIGNAL returns a set L such that $i \in L$ with probability at least $1 - O(\alpha)$. Moreover the procedure runs in expected time

$$O((\frac{B}{\alpha}\log(1/\delta) + |\operatorname{supp}(\widehat{z})| (1 + \alpha\log(1/\delta))) \log(n/B)).$$

Proof. Suppose none of $E_{coll}(i), E_{off}(i)$, and $E_{noise}(i)$ hold, as happens with probability $1 - O(\alpha)$.

Set $t = O(\log n), t' = t/3$ and $R_{loc} = O(\log_{1/\alpha}(t/\alpha))$. Let $w_0 = n/B$ and $w_D = w_0/(t')^{D-1}$, so $w_{D_{max}+1} < 1$ for $D_{max} = \log_{t'}(w_0 + 1)$. In each round D, Lemma 4.4 implies that if $\tau \in [l_j^{(D)}, l_j^{(D)} + w_D]$ then $\pi_{\sigma,b}(i) \in [l_j^{(D+1)}, l_j^{(D+1)} + w_{D+1}]$ with probability at least $1 - \alpha^{\Omega(R_{loc})} = 1 - \alpha/t$. By a union bound, with probability at least $1 - \alpha$ we have $\pi_{\sigma,b}(i) \in [l_j^{(D_{max}+1)}, l_j^{(D_{max}+1)} + w_{D_{max}+1}] = \{l_j^{(D_{max}+1)}\}$. Thus $i = \pi_{\sigma,b}^{-1}(l_j^{(D_{max}+1)}) \in L$. Since $R_{loc}D_{max} = O(\log_{1/\alpha}(t/\alpha)\log_t(n/B)) = O(\log(n/B))$, the running time is

$$O(D_{max}(R_{loc}\frac{B}{\alpha}\log(1/\delta) + R_{loc}|\operatorname{supp}(\widehat{z})|(1+\alpha\log(1/\delta)))) = O((\frac{B}{\alpha}\log(1/\delta) + |\operatorname{supp}(\widehat{z})|(1+\alpha\log(1/\delta)))\log(n/B)).$$

4.4 **Properties of ESTIMATEVALUES**

Lemma 4.6. For any $i \in L$,

$$\Pr[\left|\widehat{w}_{i} - \widehat{x'}_{i}\right|^{2} > \mu^{2}] < e^{-\Omega(R_{est})}$$

if $B > \frac{Ck}{\alpha\epsilon}$ for some constant C.

Proof. Define $e_r = \widehat{u}_j^{(r)} \omega^{a_r i} - \widehat{x'}_i$ in each round r, and $T_r = \{i' \min h_{\sigma_r, b_r}(i') = h_{\sigma_r, b_r}(i), i' \neq i\}$, and

$$\nu_i^{(r)} = \sum_{T_r} \widehat{x'}_{i'} \omega^{-a_r i'}.$$

Suppose none of $E_{coll}^{(r)}(i)$, $E_{off}^{(r)}(i)$, and $E_{noise}^{(r)}(i)$ hold, as happens with probability $1 - O(\alpha)$. Then by Lemma 4.2,

$$\mathbb{E}_{a_r}[|e_r|^2] \le 2(1+\delta)^2 \left\| \widehat{x'}_{T_r} \right\|_2^2 + O(n\delta^2)(\|x\|_2^2 + \|x'\|_2^2).$$

Hence by $E_{off}^{\left(r\right)}$ and $E_{noise}^{\left(r\right)}$ not holding,

$$\mathbb{E}_{a_r}[|e_r|^2] \le 2(1+\delta)^2 \frac{\rho^2}{\alpha B} + O(\rho^2/n^2)$$
$$\le \frac{3}{\alpha B}\rho^2 = \frac{3k}{\alpha \epsilon B}\mu^2 < \frac{3}{C}\mu^2$$

Hence with $3/4 - O(\alpha) > 5/8$ probability in total,

$$|e_r|^2 < \frac{12}{C}\mu^2 < \mu^2$$

for sufficiently large C. Thus $|\text{median}_r e_r|^2 < \mu^2$ with probability at least $1 - e^{-\Omega(R_{est})}$. Since $\widehat{w}_i = \widehat{x'}_i + \text{median} e_r$, the result follows.

Lemma 4.7. Let $R_{est} = O(\log \frac{B}{\gamma f k})$. Then if $k' = (1+f)k \le 2k$, we have $\operatorname{Err}^2(\widehat{x'_L} - \widehat{w_J}, fk) \le \operatorname{Err}^2(\widehat{x'_L}, k) + O(\epsilon)\rho^2$

with probability $1 - \gamma$.

Proof. By Lemma 4.6, each index $i \in L$ has

$$\Pr[\left|\widehat{w}_i - \widehat{x'}_i\right|^2 > \mu^2] < \frac{\gamma f k}{B}.$$

Let $U = \{i \mid \left| \widehat{w}_i - \widehat{x'}_i \right|^2 > \mu^2 \}$. With probability $1 - \gamma$, $|U| \le fk$; assume this happens. Then

$$\left\| (\widehat{x'} - \widehat{w})_{L \setminus U} \right\|_{\infty}^2 \le \mu^2.$$
(5)

Let T contain the top 2k coordinates of $\widehat{w}_{L\setminus U}$. By the analysis of Count-Sketch (most specifically, Theorem 3.1 of [PW11]), the ℓ_{∞} guarantee means that

$$\left\|\widehat{x'}_{L\setminus U} - \widehat{w}_T\right\|_2^2 \le \operatorname{Err}^2(\widehat{x'}_{L\setminus U}, k) + 3k\mu^2.$$
(6)

Because J is the top (2+f)k coordinates of $\widehat{w}, T \subset J$ and $|J \setminus T| \leq fk$. Thus

$$\begin{aligned} \operatorname{Err}^{2}(\widehat{x'_{L}} - \widehat{w_{J}}, fk) &\leq \left\| \widehat{x'_{L \setminus U}} - \widehat{w_{J \setminus U}} \right\|_{2}^{2} \\ &\leq \left\| \widehat{x'}_{L \setminus U} - \widehat{w_{T}} \right\|_{2}^{2} + \left\| (\widehat{x'} - \widehat{w})_{J \setminus (U \cup T)} \right\|_{2}^{2} \\ &\leq \left\| \widehat{x'}_{L \setminus U} - \widehat{w_{T}} \right\|_{2}^{2} + |J \setminus T| \left\| (\widehat{x'} - \widehat{w})_{J \setminus U} \right\|_{\infty}^{2} \\ &\leq \operatorname{Err}^{2}(\widehat{x'}_{L \setminus U}, k) + 3k\mu^{2} + fk\mu^{2} \\ &\leq \operatorname{Err}^{2}(\widehat{x'}_{L \setminus U}, k) + 4\epsilon\rho^{2} \end{aligned}$$

where we used Equations (5) and (6).

4.5 Properties of SPARSEFFT

Define $\hat{v}^{(r)} = \hat{x} - \hat{z}^{(r)}$. We will show that $\hat{v}^{(r)}$ gets sparser as r increases, with only a mild increase in the error.

Lemma 4.8. Consider any one loop r of SPARSEFFT, running with parameters $B = \frac{Ck}{\alpha^2 \epsilon}$ for some parameters C, f, and α , with C larger than some fixed constant. Then

$$\operatorname{Err}^{2}(\widehat{v}^{(r+1)}, 2fk) \leq (1 + O(\epsilon)) \operatorname{Err}^{2}(\widehat{v}^{(r)}, k) + O(\epsilon \delta^{2} n^{3} (\|x\|_{2}^{2} + \|\widehat{v}^{(r)}\|_{2}^{2}))$$

with probability $1 - O(\alpha/f)$, and the running time is

$$O((|\operatorname{supp}(\widehat{z}^{(r)})|(1+\alpha\log(1/\delta)) + \frac{B}{\alpha}\log(1/\delta))(\log\frac{1}{\alpha\epsilon} + \log(n/B))).$$

Proof. We use $R_{est} = O(\log \frac{B}{\alpha k}) = O(\log \frac{1}{\alpha \epsilon})$ rounds inside ESTIMATEVALUES.

The running time for LOCATESIGNAL is

$$O((\frac{B}{\alpha}\log(1/\delta) + |\operatorname{supp}(\widehat{z}^{(r)})|(1 + \alpha\log(1/\delta)))\log(n/B)),$$

and for ESTIMATEVALUES is

$$O(\log \frac{1}{\alpha \epsilon} (\frac{B}{\alpha} \log(1/\delta) + |\operatorname{supp}(\widehat{z}^{(r)})| (1 + \alpha \log(1/\delta))))$$

for a total running time as given.

Let $\mu^2 = \frac{\epsilon}{k} \operatorname{Err}^2(\widehat{v}^{(r)}, k)$, and $S = \{i \in [n] \mid \left| \widehat{v}_i^{(r)} \right|^2 > \mu^2 \}$.

By Lemma 4.5, each $i \in S$ lies in L_r with probability at least $1 - O(\alpha)$. Hence $|S \setminus L| < fk$ with probability at least $1 - O(\alpha/f)$. Let $T \subset L$ contain the largest k coordinates of $\hat{v}^{(r)}$. Then

$$\operatorname{Err}^{2}(\widehat{v}_{[n]\setminus L}^{(r)}, fk) \leq \left\|\widehat{v}_{[n]\setminus(L\cup S)}^{(r)}\right\|_{2}^{2} \leq \left\|\widehat{v}_{[n]\setminus(L\cup T)}^{(r)}\right\|_{2}^{2} + |T\setminus S| \left\|\widehat{v}_{[n]\setminus S}^{(r)}\right\|_{\infty}^{2} \leq \operatorname{Err}^{2}(\widehat{v}_{[n]\setminus L}^{(r)}, k) + k\mu^{2}.$$
 (7)

Let $\widehat{w} = \widehat{z}^{(r+1)} - \widehat{z}^{(r)} = \widehat{v}^{(r)} - \widehat{v}^{(r+1)}$ by the vector recovered by ESTIMATEVALUES. Then $\operatorname{supp}(\widehat{w}) \subset L$, so

$$\operatorname{Err}^{2}(\widehat{v}^{(r+1)}, 2fk) = \operatorname{Err}^{2}(\widehat{v}^{(r)} - \widehat{w}, 2fk)$$

$$\leq \operatorname{Err}^{2}(\widehat{v}^{(r)}_{[n] \setminus L}, fk) + \operatorname{Err}^{2}(\widehat{v}^{(r)}_{L} - \widehat{w}, fk)$$

$$\leq \operatorname{Err}^{2}(\widehat{v}^{(r)}_{[n] \setminus L}, fk) + \operatorname{Err}^{2}(\widehat{v}^{(r)}_{L}, k) + O(k\mu^{2})$$

by Lemma 4.7. But by Equation (7), this gives

$$\operatorname{Err}^{2}(\widehat{v}^{(r+1)}, 2fk) \leq \operatorname{Err}^{2}(\widehat{v}_{[n]\setminus L}^{(r)}, k) + \operatorname{Err}^{2}(\widehat{v}_{L}^{(r)}, k) + O(k\mu^{2})$$
$$\leq \operatorname{Err}^{2}(\widehat{v}^{(r)}, k) + O(k\mu^{2})$$
$$= \operatorname{Err}^{2}(\widehat{v}^{(r)}, k) + O(\epsilon\rho^{2}).$$

The result follows from the definition of ρ^2 .

Given the above, this next proof largely follows the argument of [IPW11], Theorem 3.7.

Theorem 4.9. SPARSEFFT recovers $\hat{z}^{(R+1)}$ with

$$\left\|\widehat{x} - \widehat{z}^{(R+1)}\right\|_{2} \le (1+\epsilon)\operatorname{Err}(\widehat{x}, k) + \delta \|\widehat{x}\|_{2}$$

in $O(\frac{k}{\epsilon}\log(n/k)\log(n/\delta))$ time.

Proof. Define $f_r = O(1/r^2)$ so $\sum f_r < 1/4$. Choose R so $\prod_{r \le R} f_r < 1/k \le \prod_{r < R} f_r$. Then $R = O(\log k / \log \log k)$, since $\prod_{r \le R} f_r < f_{R/2}^{R/2} = (2/R)^R$.

Set $\epsilon_r = f_r \epsilon$, $\alpha_r = \Theta(f_r^2)$, $k_r = k \prod_{i < r} f_i$, $B_r = O(\frac{k}{\epsilon} \alpha_r f_r)$. Then $B_r = \omega(\frac{k_r}{\alpha_r^2 \epsilon_r})$, so for sufficiently large constant the constraint of Lemma 4.8 is satisfied. For appropriate constants, Lemma 4.8 says that in each round r,

$$\operatorname{Err}^{2}(\widehat{v}^{(r+1)}, k_{r+1}) = \operatorname{Err}^{2}(\widehat{v}^{(r+1)}, f_{r}k_{r}) \leq (1 + f_{r}\epsilon)\operatorname{Err}^{2}(\widehat{v}^{(r)}, k_{r}) + O(f_{r}\epsilon\delta^{2}n^{3}(\|x\|_{2}^{2} + \|\widehat{v}^{(r)}\|_{2}^{2}))$$
(8)

with probability at least $1 - f_r$. Now, the change $\widehat{w} = \widehat{v}^{(r)} - \widehat{v}^{(r+1)}$ in round r is a median of HASHTOBINS results \widehat{u} . Hence by Lemma 3.3,

$$\begin{split} \|\widehat{w}\|_{1} &\leq 2 \max \|\widehat{u}\|_{1} \leq 2((1+\delta) \left\|\widehat{v}^{(r)}\right\|_{1} + \delta n(\|x\|_{1} + 2 \left\|\widehat{x} - \widehat{v}^{(r)}\right\|_{1})) \\ & \left\|\widehat{v}^{(r+1)}\right\|_{1} \leq 3 \left\|\widehat{v}^{(r)}\right\|_{1} + O(\delta n)(\sqrt{n} \|\widehat{x}\|_{2} + \left\|\widehat{v}^{(r)}\right\|_{2}) \\ & \leq 3 \left\|\widehat{v}^{(r)}\right\|_{1} + O(\delta n\sqrt{n})(\|\widehat{x}\|_{1} + \left\|\widehat{v}^{(r)}\right\|_{1}) \end{split}$$

We shall show by induction that $\|\hat{v}^{(r)}\|_1 \leq 4^{r-1} \|\hat{x}\|_1$. It is true for r = 1, and then since $r \leq R < \log k$,

$$\begin{split} \left\| \widehat{v}^{(r+1)} \right\|_{1} &\leq 3 \left\| \widehat{v}^{(r)} \right\|_{1} + O(\delta n \sqrt{n}) (\|\widehat{x}\|_{1} + 4^{r-1} \|\widehat{x}\|_{1}) \\ &\leq 3 \left\| \widehat{v}^{(r)} \right\|_{1} + O(\delta n \sqrt{n} k \|\widehat{x}\|_{1}) \leq 4 \left\| \widehat{v}^{(r)} \right\|_{1}. \end{split}$$

Therefore $\|\hat{v}^{(r)}\|_{2}^{2} \le 4^{r} \|\hat{x}\|_{1} \le k \|\hat{x}\|_{1} \le n\sqrt{n} \|\hat{x}\|_{2}$. Plugging into Equation (8),

$$\operatorname{Err}^{2}(\hat{v}^{(r+1)}, k_{r+1}) \leq (1 + f_{r}\epsilon) \operatorname{Err}^{2}(\hat{v}^{(r)}, k_{r}) + O(f_{r}\epsilon\delta^{2}n^{4.5} \|x\|_{2}^{2})$$

with probability at least $1 - f_r$. The error accumulates, so in round r we have

$$\operatorname{Err}^{2}(\widehat{v}^{(r)}, k_{r} \prod_{i < r} f_{i}) \leq \operatorname{Err}^{2}(\widehat{x}, k) \prod_{i < r} (1 + f_{i}\epsilon) + \sum_{i < r} O(f_{r}\epsilon\delta^{2}n^{4.5} \|x\|_{2}^{2}) \prod_{i < j < r} (1 + f_{j}\epsilon)$$

with probability at least $1 - \sum_{i < r} f_i > 3/4$. Hence in the end, since $k \prod_{i < r} f_i < 1$,

$$\left\| \widehat{v}^{(R+1)} \right\|_{2}^{2} = \operatorname{Err}^{2}(\widehat{v}^{(R+1)}, 1 - o(1)) \le \operatorname{Err}^{2}(\widehat{x}, k) \prod_{i \le R} (1 + f_{i}\epsilon) + O(R\epsilon\delta^{2}n^{4.5} \|x\|_{2}^{2}) \prod_{i < R} (1 + f_{i}\epsilon)$$

with probability at least 3/4. We also have

$$\prod_{i} (1 + f_i \epsilon) \le e^{\epsilon \sum_i f_i} \le e$$

making

$$\prod_{i} (1+f_i\epsilon) \le 1 + e \sum_{i} f_i\epsilon < 1 + 2\epsilon.$$

Thus we get the approximation factor

$$\left\|\widehat{x} - \widehat{z}^{(R+1)}\right\|_{2}^{2} \le (1+2\epsilon) \operatorname{Err}^{2}(\widehat{x},k) + O((\log k)\delta^{2}n^{4.5} \|x\|_{2}^{2})$$

with at least 3/4 probability. Rescaling δ by poly(n) and taking the square root gives the desired

$$\left\| \widehat{x} - \widehat{z}^{(R+1)} \right\|_2 \le (1+\epsilon) \operatorname{Err}(\widehat{x}, k) + \delta \|x\|_2.$$

Now we analyze the running time. The update $\hat{z}^{(r+1)} - \hat{z}^{(r)}$ in round r has support size $2k_r$, so in round r

$$|\operatorname{supp}(\widehat{z}^{(r)})| \le \sum_{i < r} 2k_r = O(k).$$

Thus the expected running time in round r is (recalling that we replaced δ by $\delta/n^{O(1)})$

$$O(\left(\left|\operatorname{supp}(\widehat{z}^{(r)})\right|(1+\alpha_r\log(n/\delta)) + \frac{B_r}{\alpha_r}\log(n/\delta))(\log\frac{1}{\alpha_r\epsilon_r} + \log(n/B_r)))$$
$$=O(\left(k+\frac{k}{r^4}\log(n/\delta) + \frac{k}{\epsilon r^2}\log(n/\delta)\right)(\log r + \log\frac{1}{\epsilon} + \log(n\epsilon/k) + \log r))$$
$$=O(\left(k+\frac{k}{\epsilon r^2}\log(n/\delta)\right)(\log r + \log(n/k)))$$

We split the terms multiplying k and $\frac{k}{\epsilon r^2} \log(n/\delta)$, and sum over r. First,

$$\sum_{r=1}^{R} (\log r + \log(n/k)) \leq O(R \log R + R \log(n/k))$$
$$\leq O(\log k + \log k \log(n/k))$$
$$= O(\log k \log(n/k)).$$

Next,

$$\sum_{r=1}^{R} \frac{1}{r^2} (\log r + \log(n/k)) = O(\log(n/k))$$

Thus the total running time is

$$O(k\log k\log(n/k) + \frac{k}{\epsilon}\log(n/\delta)\log(n/k)) = O(\frac{k}{\epsilon}\log(n/\delta)\log(n/k)).$$

5 Reducing the full k-dimensional DFT to the exact k-sparse case in n dimensions

In this section we show the following lemma. Assume that k divides n.

Lemma 5.1. Suppose that there is an algorithm A that given a vector y such that \hat{y} is k-sparse, computes \hat{y} in time T(k). Then there is an algorithm A' that given a k-dimensional vector x computes \hat{x} in time O(T(k)).

Proof. Given a k-dimensional vector x, we define $y_i = x_i \mod k$, for $i = 0 \dots n - 1$. Whenever A requests a sample y_i , we compute it from x in constant time. Moreover, we have that $\hat{y}_i = \hat{x}_{i/(n/k)}$ if i divides (n/k), and $\hat{y}_i = 0$ otherwise. Thus \hat{y} is k-sparse. Since \hat{x} can be immediately recovered from \hat{y} , the lemma follows.

Corollary 5.2. Assume that the *n*-dimensional DFT cannot be computed in $o(n \log n)$ time. Then any algorithm for the k-sparse DFT (for vectors of arbitrary dimension) must run in $\Omega(k \log k)$ time.

6 Lower Bound

In this section, we show any algorithm satisfying (1) must access $\Omega(k \log(n/k)/\log \log n)$ samples of x.

We translate this problem into the language of compressive sensing:

Theorem 6.1. Let $F \in \mathbb{C}^{n \times n}$ be orthonormal and satisfy $|F_{i,j}| = 1/\sqrt{n}$ for all i, j. Suppose an algorithm takes m adaptive samples of Fx and computes x^* with

$$||x - x^*||_2 \le 2 \min_{k \text{-sparse } x'} ||x - x'||_2$$

for any x, with probability at least 3/4. Then $m = \Omega(k \log(n/k) / \log \log n)$.

Corollary 6.2. Any algorithm computing the approximate Fourier transform must access $\Omega(k \log(n/k)/\log \log n)$ samples from the time domain.

If the samples were chosen non-adaptively, we would immediately have $m = \Omega(k \log(n/k))$ by [PW11]. However, an algorithm could choose samples based on the values of previous samples. In the sparse recovery framework allowing general linear measurements, this adaptivity can decrease the number of measurements to $O(k \log \log(n/k))$ [IPW11]; in this section, we show that adaptivity is much less effective in our setting where adaptivity only allows the choice of Fourier coefficients.

We follow the framework of Section 4 of [PW11]. Let $\mathcal{F} \subset \{S \subset [n] \mid |S| = k\}$ be a family of k-sparse supports such that:

- $|S \oplus S'| \ge k$ for $S \ne S' \in \mathcal{F}$, where \oplus denotes the exclusive difference between two sets,
- $\Pr_{S \in \mathcal{F}}[i \in S] = k/n$ for all $i \in [n]$, and
- $\log |\mathcal{F}| = \Omega(k \log(n/k)).$

This is possible; for example, a random linear code on $[n/k]^k$ with relative distance 1/2 has these properties.¹⁰

For each $S \in \mathcal{F}$, let $X^S = \{x \in \{0, \pm 1\}^n \mid \operatorname{supp}(x^S) = S\}$. Let $x \in X^S$ uniformly at random. The variables $x_i, i \in S$, are i.i.d. subgaussian random random variables with parameter $\sigma^2 = 1$, so for any row F_i of F, $F_j x$ is subgaussian with parameter $\sigma^2 = k/n$. Therefore

$$\Pr_{x \in X^S}[|F_j x| > t\sqrt{k/n}] < 2e^{-t^2/2}$$

hence there exists an $x^S \in X^S$ with

$$\left\|Fx^{S}\right\|_{\infty} < O(\sqrt{\frac{k\log n}{n}}).$$
⁽⁹⁾

Let $X = \{x^S \mid S \in \mathcal{F}\}$ be the set of all such x^S .

Let $w \sim N(0, \alpha \frac{k}{n} I_n)$ be i.i.d. normal with variance $\alpha k/n$ in each coordinate.

Consider the following process:

¹⁰This assumes n/k is a prime larger than 2. If n/k is not prime, we can choose $n' \in [n/2, n]$ to be a prime multiple of k, and restrict to the first n' coordinates. This works unless n/k < 3, in which case the bound of $\Theta(k \log(n/k)) = \Theta(k)$ is trivial.

Procedure First, Alice chooses $S \in \mathcal{F}$ uniformly at random, then $x \in X$ subject to $\operatorname{supp}(x) = S$, then $w \sim N(0, \alpha \frac{k}{n}I_n)$ for $\alpha = \Theta(1)$. For $j \in [m]$, Bob chooses $i_j \in [n]$ and observes $y_j = F_{i_j}(x+w)$. He then computes the result $x' \approx x$ of sparse recovery, rounds to X by $\hat{x} = \arg \min_{x^* \in X} ||x^* - x'||_2$, and sets $S' = \operatorname{supp}(\hat{x})$. This gives a Markov chain $S \to x \to y \to x' \to \hat{x} \to S'$.

We will show that deterministic sparse recovery algorithms require large m to succeed on this input distribution x + w with 3/4 probability. As a result, randomized sparse recovery algorithms require large m to succeed with 3/4 probability.

Our strategy is to give upper and lower bounds on I(S; S'), the mutual information between S and S'.

Lemma 6.3 (Analog of Lemma 4.3 of [PW11] for $\epsilon = O(1)$). There exists a constant $\alpha' > 0$ such that if $\alpha < \alpha'$, then $I(S; S') = \Omega(k \log(n/k))$.

Proof. Assuming the sparse recovery succeeds (as happens with 3/4 probability), we have $||x' - (x + w)||_2 \le 2 ||w||_2$, which implies $||x' - x||_2 \le 3 ||w||_2$. Therefore

$$\begin{aligned} \|\hat{x} - x\|_2 &\leq \left\|\hat{x} - x'\right\|_2 + \left\|x' - x\right\|_2 \\ &\leq 2 \left\|x' - x\right\|_2 \\ &\leq 6 \|w\|_2. \end{aligned}$$

We also know $||x' - x''||_2 \ge \sqrt{k}$ for all distinct $x', x'' \in X$ by construction. With probability at least 3/4 we have $||w||_2 \le \sqrt{4\alpha k} < \sqrt{k}/6$ for sufficiently small α . But then $||\hat{x} - x||_2 < \sqrt{k}$, so $\hat{x} = x$ and S = S'. Thus $\Pr[S \neq S'] \le 1/2$.

Fano's inequality states $H(S \mid S') \leq 1 + \Pr[S \neq S'] \log |\mathcal{F}|$. Thus

$$I(S; S') = H(S) - H(S \mid S') \ge -1 + \frac{1}{2} \log |\mathcal{F}| = \Omega(k \log(n/k))$$

as desired.

We next show an analog of their upper bound (Lemma 4.1 of [PW11]) on I(S; S') for adaptive measurements of bounded ℓ_{∞} norm. The proof follows the lines of [PW11], but is more careful about dependencies and needs the ℓ_{∞} bound on Fx.

Lemma 6.4.

$$I(S; S') \le O(m \log(1 + \frac{1}{\alpha} \log n)).$$

Proof. Let $A_j = F_{i_j}$ for $j \in [m]$, and let $w'_j = F_{i_j}w$. The w'_j are independent normal variables with variance $\alpha \frac{k}{n}$.

Let $y_j \stackrel{''}{=} A_j x + w'_j$. We know $I(S; S') \leq I(x; y)$ because $S \to x \to y \to S'$ is a Markov chain. Because the variables A_j are deterministic given y_1, \ldots, y_{j-1} , we have by the chain rule for information

that

$$\begin{split} I(S;S') &\leq I(x;y) \\ &= I(x;y_1) + \sum_{j=2}^{m} I(x;y_j \mid y_1, \dots, y_{j-1}) \\ &\leq I(A_1x;y_1) + \sum_{j=2}^{m} I(A_jx;y_j \mid y_1, \dots, y_{j-1}) \\ &= I(A_1x;A_1x + w_1') + \sum_{j=2}^{m} I(A_jx;A_jx + w_j' \mid y_1, \dots, y_{j-1}) \\ &= H(A_1x + w_1') - H(A_1x + w_1' \mid A_1x) + \sum_{j=2}^{m} H(A_jx + w_j' \mid y_1, \dots, y_{j-1}) - H(A_jx + w_j' \mid A_jx, y_1, \dots, y_{j-1}) \\ &= H(A_1x + w_1') - H(w_1' \mid A_1x) + \sum_{j=2}^{m} H(A_jx + w_j' \mid y_1, \dots, y_{j-1}) - H(w_j' \mid A_jx, y_1, \dots, y_{j-1}) \\ &= H(A_1x + w_1' \mid A_1) - H(w_1' \mid A_1x, A_1) + \sum_{j=2}^{m} H(A_jx + w_j' \mid y_1, \dots, y_{j-1}, A_j) - H(w_j' \mid A_jx, A_j) \\ &\leq H(A_1x + w_1' \mid A_1) - H(w_1' \mid A_1x, A_1) + \sum_{j=2}^{m} H(A_jx + w_j' \mid A_j) - H(w_j' \mid A_jx, A_j) \\ &= H(A_1x + w_1' \mid A_1) - H(w_1' \mid A_1x, A_1) + \sum_{j=2}^{m} H(A_jx + w_j' \mid A_j) - H(w_j' \mid A_jx, A_j) \\ &= H(A_1x + w_1' \mid A_1) - H(A_1x + w_1' \mid A_1x, A_1) + \sum_{j=2}^{m} H(A_jx + w_j' \mid A_j) - H(A_jx + w_j' \mid A_jx, A_j) \\ &= \sum_j I(A_jx; A_jx + w_j' \mid A_j). \end{split}$$

Thus it suffices to show $I(A_j x; A_j x + w'_j \mid A_j) = O(\log(1 + \frac{1}{\alpha} \log n))$ for all j. We have

$$I(A_j x; A_j x + w'_j \mid A_j) = \mathbb{E}_{A_j}[I(A_j x; A_j x + w'_j)]$$

Note that A_j is a row of F and $w'_j \sim N(0, \frac{\alpha k}{n})$ independently. Hence it suffices to show that for any row v of F, for $u \sim N(0, \frac{\alpha k}{n})$ we have

$$I(vx; vx + u) = O(\log(1 + \frac{1}{\alpha}\log n)).$$

But we know $|vx| \le O(\sqrt{\frac{k \log n}{n}})$ by Equation (9). By the Shannon-Hartley theorem on channel capacity of Gaussian channels under a power constraint,

$$I(vx; vx + u) \leq \frac{1}{2}\log(1 + \frac{\mathbb{E}[(vx)^2]}{\mathbb{E}[u^2]})$$
$$= \frac{1}{2}\log(1 + \frac{n}{\alpha k}O(\frac{k\log n}{n}))$$
$$= O(\log(1 + \frac{1}{\alpha}\log n))$$

as desired.

Theorem 6.1 follows from Lemma 6.3 and Lemma 6.4, with $\alpha = \Theta(1)$.

7 Efficient Constructions of Window Functions

Claim 7.1. Let cdf denote the standard Gaussian cumulative distribution function. Then:

1.
$$\operatorname{cdf}(t) = 1 - \operatorname{cdf}(-t).$$

2. $\operatorname{cdf}(t) \le e^{-t^2/2} \text{ for } t < 0.$
3. $\operatorname{cdf}(t) < \delta \text{ for } t < -\sqrt{2\log(1/\delta)}.$
4. $\int_{x=-\infty}^{t} \operatorname{cdf}(x) dx < \delta \text{ for } t < -\sqrt{4\pi \log(1/\delta)}.$
5. For any δ , there exists a function $\operatorname{cdf}_{\delta}(t)$ computable in $O(\log(1/\delta))$ time such that $\left\|\operatorname{cdf}-\operatorname{cdf}_{\delta}\right\|_{\infty} < \delta$

Proof.

- 1. Follows from the symmetry of Gaussian distribution.
- 2. Follows from a standard moment generating function bound on Gaussian random variables.
- 3. Follows from (2).
- 4. Property (2) implies that cdf(t) is at most $\sqrt{2\pi}$ larger than the Gaussian pdf. Then apply (3).
- 5. By (1) and (3), $\operatorname{cdf}(t)$ can be computed as $\pm \delta$ or $1 \pm \delta$ unless $|t| < \sqrt{2(\log(1/\delta))}$. But then an efficient expansion around 0 only requires $O(\log(1/\delta))$ terms to achieve precision $\pm \delta$.

For example, we can truncate the representation [Mar04]

$$\operatorname{cdf}(t) = \frac{1}{2} + \frac{e^{-t^2/2}}{\sqrt{2\pi}} \left(t + \frac{t^3}{3} + \frac{t^5}{3 \cdot 5} + \frac{t^7}{3 \cdot 5 \cdot 7} + \cdots \right)$$

at $O(\log(1/\delta))$ terms.

Claim 7.2. Define the continuous Fourier transform of f(t) by

$$\widehat{f}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t) dt.$$

For $t \in [n]$, define

$$g_t = \sum_{j=-\infty}^{\infty} f(t+nj)$$

and

$$g'_t = \sum_{j=-\infty}^{\infty} \widehat{f}(t/n+j).$$

Then $\hat{g} = g'$, where \hat{g} is the *n*-dimensional DFT of *g*.

Proof. Let $\Delta_1(t)$ denote the Dirac comb of period 1: $\Delta_1(t)$ is a Dirac delta function when t is an integer zero elsewhere. Then $\widehat{\Delta_1} = \Delta_1$. For any $t \in [n]$, we have

$$\hat{g}_{t} = \sum_{s=1}^{n} \sum_{j=-\infty}^{\infty} f(s+nj)e^{-2\pi i t s/n}$$

$$= \sum_{s=1}^{n} \sum_{j=-\infty}^{\infty} f(s+nj)e^{-2\pi i t (s+nj)/n}$$

$$= \sum_{s=-\infty}^{\infty} f(s)e^{-2\pi i t s/n}$$

$$= \int_{-\infty}^{\infty} f(s)\Delta_{1}(s)e^{-2\pi i t s/n} ds$$

$$= (\widehat{f} \cdot \widehat{\Delta_{1}})(t/n)$$

$$= (\widehat{f} * \Delta_{1})(t/n)$$

$$= \sum_{j=-\infty}^{\infty} \widehat{f}(t/n+j)$$

$$= g'_{t}.$$

Lemma 7.3. There exist flat window functions G and $\widehat{G'}$ with parameters b, δ , and α such that G can be computed in $O(\frac{B}{\alpha}\log(1/\delta))$ time, and for each $i \widehat{G'}_i$ can be evaluated in $O(\log(1/\delta))$ time.

Proof. We will show this for a function $\widehat{G'}$ that is (approximately) a Gaussian convolved with a box-car filter. First we construct analogous window functions for the continuous Fourier transform. We then show that discretizing these functions gives the desired result.

Let D be a Gaussian with standard deviation σ to be determined later, so \widehat{D} is a Gaussian with standard deviation $1/\sigma$. Let \widehat{F} be a box-car filter of length 2C for some parameter C; that is, let $\widehat{F}(t) = 1$ for |t| < C and F(t) = 0 otherwise, so $F(t) = \operatorname{sinc}(t/C)$. Let $G^* = D \cdot F$, so $\widehat{G^*} = \widehat{D} * \widehat{F}$.

Then $|G^*(t)| \leq |D(t)| < \delta$ for $t > \sigma \sqrt{2 \log(1/\delta)}$. Furthermore, G^* is computable in O(1) time. Its Fourier transform is $\widehat{G^*}(t) = \operatorname{cdf}(\sigma(t+C)) - \operatorname{cdf}(\sigma(t-C))$. By Claim 7.1 we have for $|t| > C + \sqrt{2 \log(1/\delta)}/\sigma$ that $\widehat{G^*}(t) = \pm \delta$. We also have, for $|t| < C - \sqrt{2 \log(1/\delta)}/\sigma$, that $\widehat{G^*}(t) = 1 \pm 2\delta$.

 $C + \sqrt{2\log(1/\delta)}/\sigma \operatorname{that} \widehat{G^*}(t) = \pm \delta. \text{ We also have, for } |t| < C - \sqrt{2\log(1/\delta)}/\sigma, \operatorname{that} \widehat{G^*}(t) = 1 \pm 2\delta.$ Now, for $i \in [n]$ let $H_i = \sum_{j=\infty}^{\infty} G^*(i+nj)$. By Claim 7.2 it has DFT $\widehat{H}_i = \sum_{j=\infty}^{\infty} \widehat{G^*}(i/n+j)$. Furthermore, $\sum_{|i| > \sigma\sqrt{2\log(1/\delta)}} |G^*(i)| \le 2\operatorname{cdf}(-\sqrt{2\log(1/\delta)}) \le 2\delta.$

Similarly, from Claim 7.1, property (4), we have that if $1/2 > C + \sqrt{4\pi \log(1/\delta)} / \sigma$ then $\sum_{|i|>n/2} \left|\widehat{G^*}(i/n)\right| \le 4\delta$. Then for any $|i| \le n/2$, $\widehat{H}_i = \widehat{G^*}(i/n) \pm 4\delta$.

Let

$$G_i = \sum_{\substack{|j| < \sigma \sqrt{2\log(1/\delta)} \\ j \equiv i \pmod{n}}} G^*(j)$$

for $|i| < \sigma \sqrt{2 \log(1/\delta)}$ and $G_i = 0$ otherwise. Then $||G - H||_1 \le 2\delta$. Let

$$\widehat{G'}_{i} = \begin{cases} 1 & |i| \le n(C - \sqrt{2\log(1/\delta)}/\sigma) \\ 0 & |i| \ge n(C + \sqrt{2\log(1/\delta)}/\sigma) \\ \widetilde{\operatorname{cdf}}_{\delta}(\sigma(i+C)/n) - \widetilde{\operatorname{cdf}}_{\delta}(\sigma(i-C)/n) & \text{otherwise} \end{cases}$$

where $\widetilde{\operatorname{cdf}}_{\delta}(t)$ computes $\operatorname{cdf}(t)$ to precision $\pm \delta$ in $O(\log(1/\delta))$ time, as per Claim 7.1. Then $\widehat{G'}_i = \widehat{G^*}(i/n) \pm 2\delta = \widehat{H}_i \pm 6\delta$. Hence

$$\left\|\widehat{G}-\widehat{G'}\right\|_{\infty} \leq \left\|\widehat{G'}-\widehat{H}\right\|_{\infty} + \left\|\widehat{G}-\widehat{H}\right\|_{\infty} \leq \left\|\widehat{G'}-\widehat{H}\right\|_{\infty} + \left\|\widehat{G}-\widehat{H}\right\|_{2} = \left\|\widehat{G'}-\widehat{H}\right\|_{\infty} + \left\|G-H\right\|_{2} \leq 6\delta + 2\delta = 8\delta$$

Replacing δ by $\delta/8$ and plugging in $\sigma = \frac{4B}{\alpha}\sqrt{2\log(1/\delta)}$ and $C = (1 - \alpha/2)/(2B)$, we have that:

- $|G_i| = 0$ for $|i| \ge \Omega(\frac{B}{\alpha}\log(1/\delta))$
- $\widehat{G'}_i = 1$ for $|i| \le (1 \alpha)n/(2B)$
- $\widehat{G'}_i = 0$ for $|i| \ge n/(2B)$
- $\widehat{G'}_i \in [0,1]$ for all i.

•
$$\left\|\widehat{G'} - \widehat{G}\right\|_{\infty} < \delta.$$

- We can compute G over its entire support in $O(\frac{B}{\alpha}\log(n/\delta))$ total time.
- For any i, $\widehat{G'}_i$ can be computed in $O(\log(1/\delta))$ time for $|i| \in [(1 \alpha)n/(2B), n/(2B)]$ and O(1) time otherwise.

We needed that $1/2 \ge (1 - \alpha/2)/(2B) + \sqrt{2\pi}\alpha/(4B)$, which holds for $B \ge 2$. The B = 1 case is trivial, using the constant function $\widehat{G'}_i = 1$.

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