

Define the language L as:

$$L = \{0^i 1^i 2^i \mid i \geq 1\}$$

Theorem. L is not a context free language.

Proof. (By contradiction) Suppose that L is a context free language.

Let G be a CNF context-free grammar with k nonterminals such that $L = L(G)$, and let $n = 2^k$.

Consider the string $z = 0^n 1^n 2^n$. Clearly, $z \in L$, and $|z| = 3n$.

Since $|z| \geq n$, it follows from the pumping lemma that z can be broken up into five parts i.e., $z = uvwxy$, such that $|vx| \geq 1$, $|vwx| \leq n$, and $uv^i wx^i y \in L(G)$, for all $i \geq 0$.

Consider which parts of $z = 0^n 1^n 2^n$ form the substrings v and x .

Case 1) vx contains only 0's.

Then consider the string $z' = uv^2wx^2y$. By the pumping lemma $z' \in L(G)$. But since $|vx| \geq 1$, and v and x contain only 0's, it follows that z' is of the form $0^m 1^n 2^n$, where $m > n$. Hence, $z' \notin L$, a contradiction. Similarly if vx contains only 1's, or if it contains only 2's.

Case 2) vx contains both 0's and 1's, but no 2's.

Then consider the string $z' = uv^2wx^2y$. By the pumping lemma $z' \in L(G)$. But since $|vx| \geq 1$, and v and x contain 0's and 1's, it follows that z' contains more 0's and 1's than 2's. Hence, $z' \notin L$, a contradiction. Similarly if vx contains both 1's and 2's, but not 0's.

Case 3) vx contains both 0's and 2's.

Can't happen since $z' = 0^n 1^n 2^n$ and $|vwx| \leq n$. In other words, v and x can contain at most two different symbols. Furthermore, if they do contain two different symbols, than those symbols must be consecutive. \square