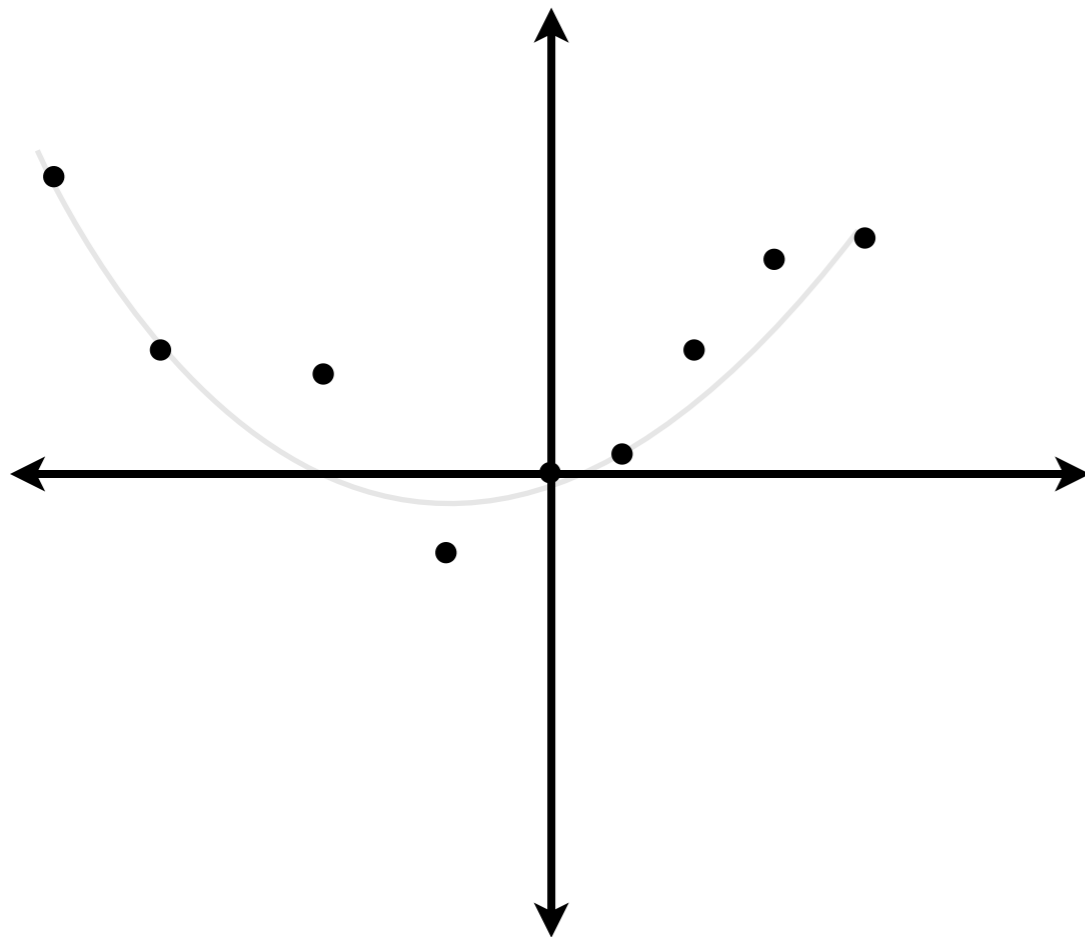


Radial Basis Function Interpolation

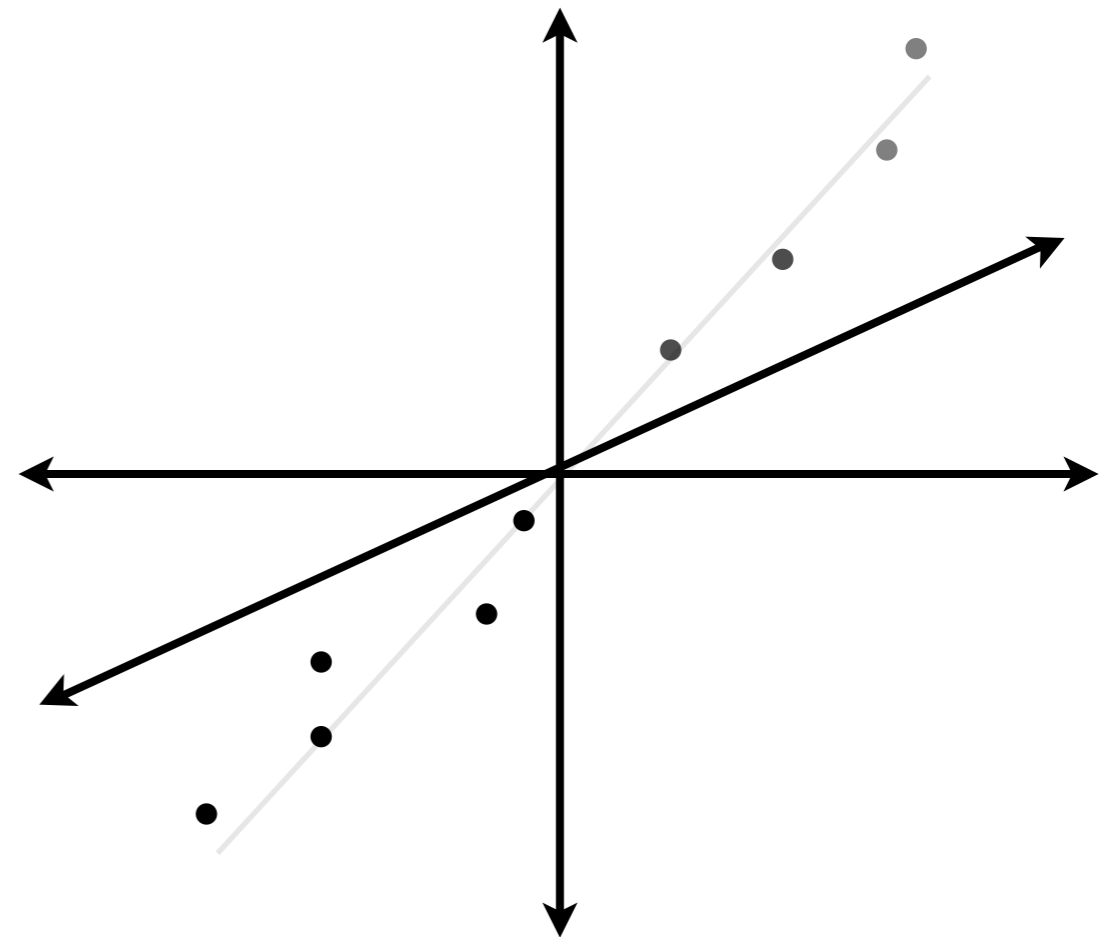
Kim Day
Jessie Twigger

Before

- Worked in 2D or 3D space



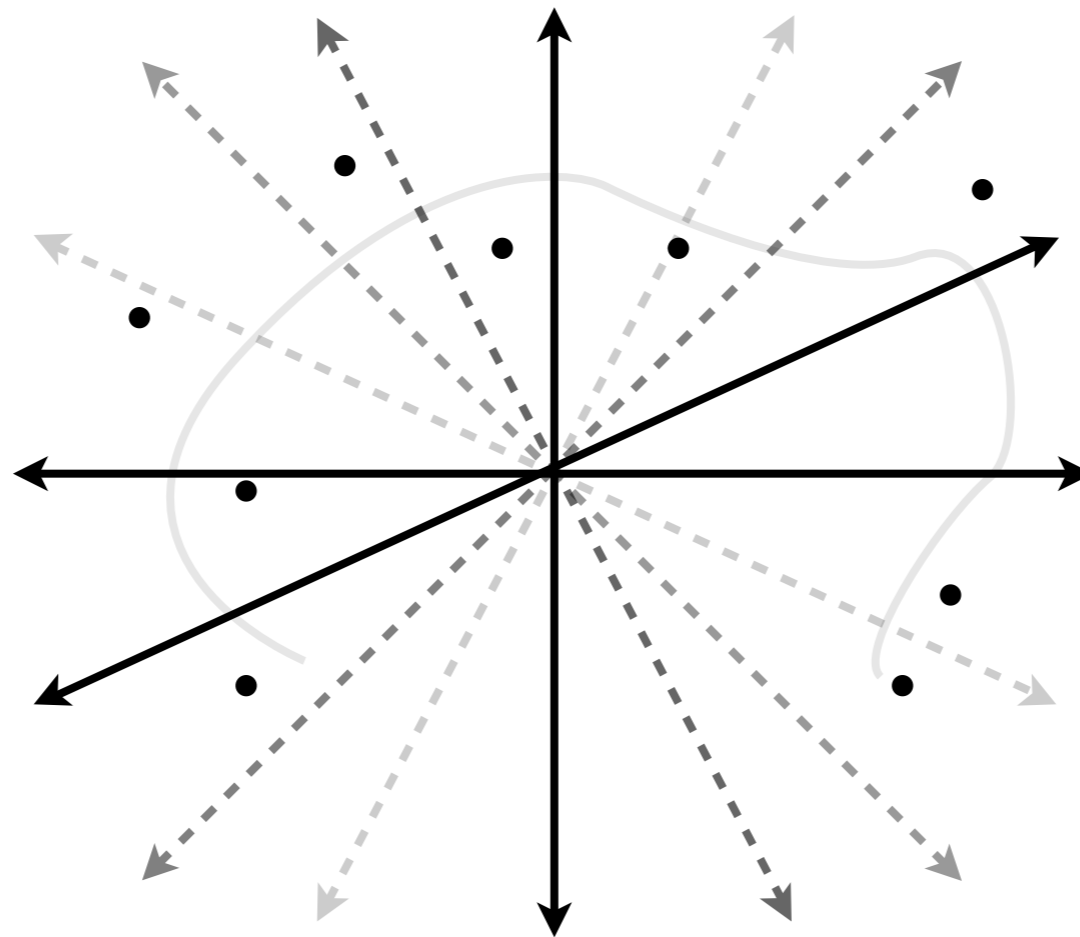
Example of 2D interpolation



Example of 3D interpolation

Stepping up the dimension

- Can work in nD space now!

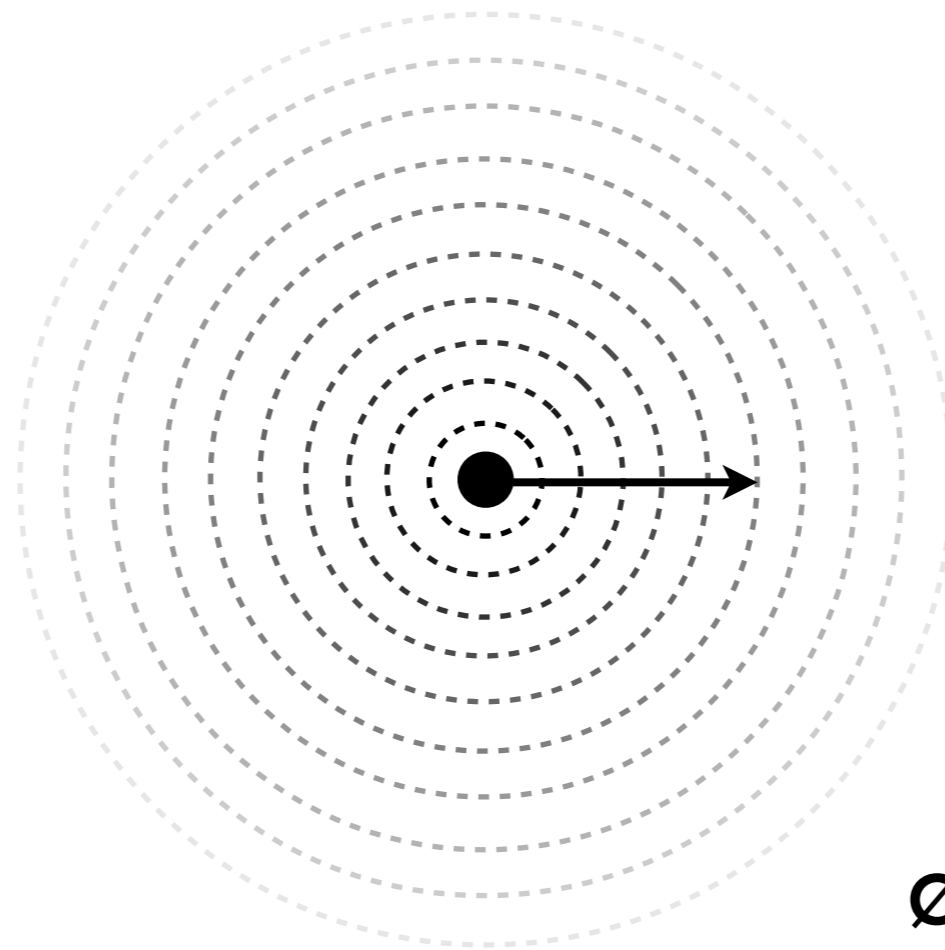


Terrible example of 8D interpolation

Definitions

Radial Basis Function

- Depends only on the *distance* from a point

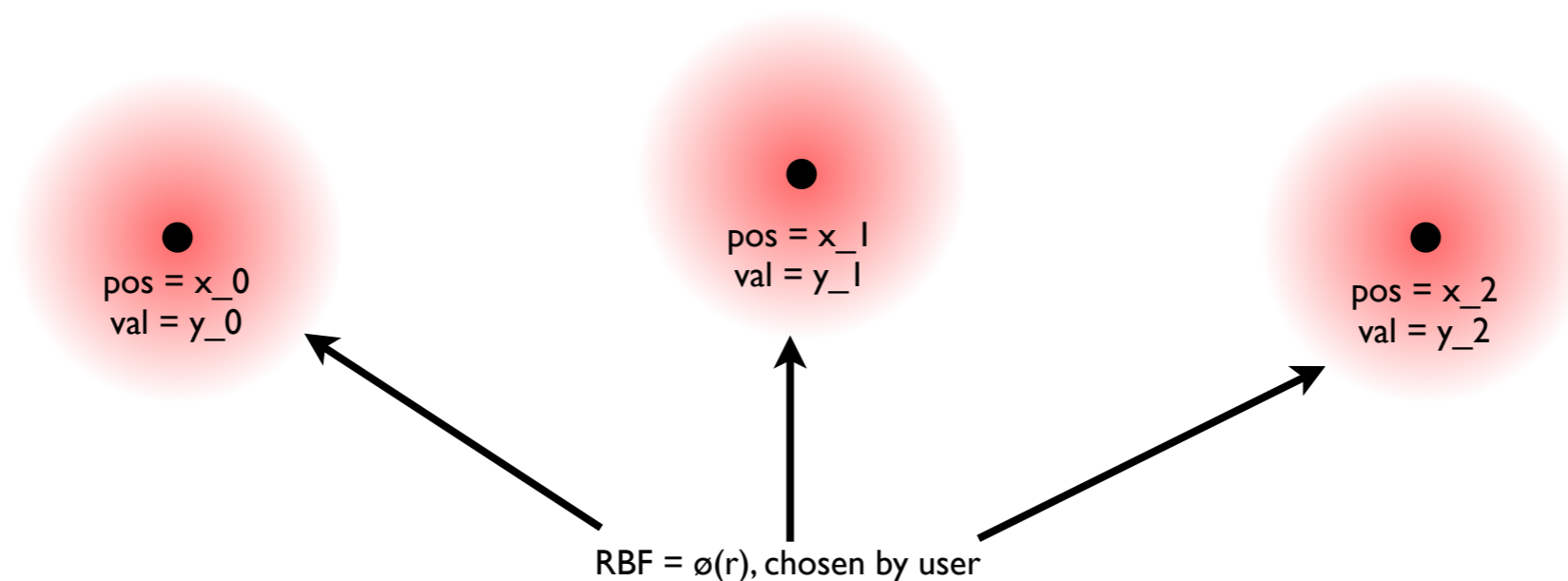


$$\phi(\mathbf{x}) = \phi(\|\mathbf{x}\|)$$

Description

RBF Interpolation

- Imagine that every point in the series has a field around it (an RBF).
- Each point has a position x_i and value y_i .
- ($x_i =$ vector in nD space)

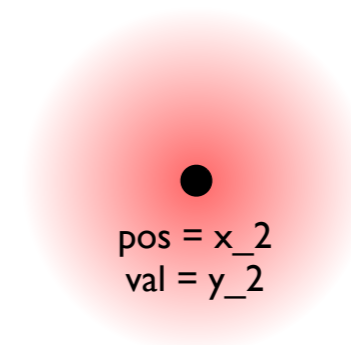
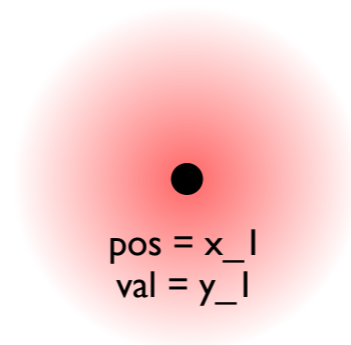
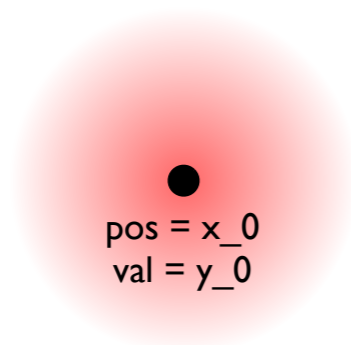


Example of three points in nD space

RBF Interpolation

- Want to find the y value at an arbitrary point in this nD space.

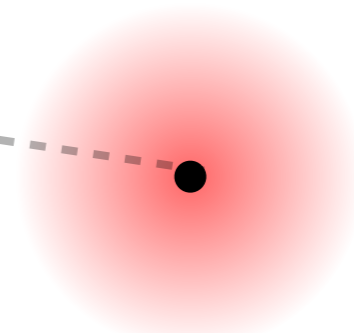
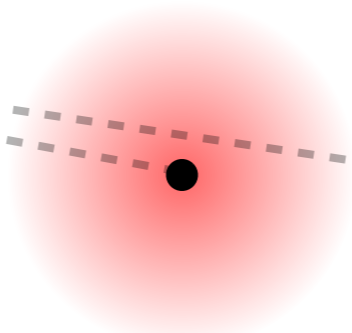
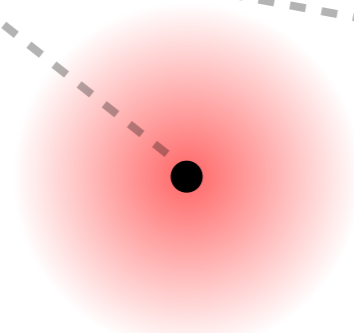
pos = x
val = $y = ?$



Example of three points in nD space with arbitrary point (star)

RBF Interpolation

- Can approximate the “field strength” at this point by adding a linear combination of the RBFs.
- Just need to find the w_i coefficients.



$$y(\vec{x}) = \sum_{i=0}^{N-1} w_i \phi(|\vec{x} - \vec{x}_i|)$$

Evaluating the radial functions

Finding w_i

- Finding the w_i coefficients is a matter of solving a system of linear equations.
- Use the known y_i values from the N points to create a system of N linear equations and N unknowns.
- Then, use linear equation solving method of choice to solve.

$$y(\vec{x}) = \sum_{i=0}^{N-1} w_i \phi(|\vec{x} - \vec{x}_i|)$$

RBF Types

Name	Equation	Notes
Multiquadratic	$\phi(r) = (r^2 + r_0^2)^{\frac{1}{2}}$	Most common
Inverse multiquadratic	$\phi(r) = (r^2 + r_0^2)^{-\frac{1}{2}}$	Sometimes better than multiquadratic
Thin-plate spine	$\phi(r) = r^2 \log \frac{r}{r_0}$	Ideal for thin plate warping problem, otherwise just okay
Gaussian	$\phi(r) = \exp\left(-\frac{1}{2} \frac{r^2}{r_0^2}\right)$	Difficult but possible to achieve high accuracy; goes to zero quickly

Choosing r_0

- Arbitrary scale factor for RBF
- Sensitivity to r_0 varies from function to function
- Should be larger than average point separation, but smaller than feature size of function

(from the previous slide)

$$\phi(r) = (r^2 + r_0^2)^{\frac{1}{2}} \quad \phi(r) = (r^2 + r_0^2)^{-\frac{1}{2}} \quad \phi(r) = r^2 \log \frac{r}{r_0} \quad \phi(r) = \exp\left(-\frac{1}{2} \frac{r^2}{r_0^2}\right)$$

Normalized RBF (NRBF)

- We require the sum of the basis functions to be unity

- $$y(x) = \frac{\sum_{i=0}^{N-1} w_i \varphi(|x-x_i|)}{\sum_{i=0}^{N-1} \varphi(|x-x_i|)} \quad (3.7.3)$$

- $$y_j \sum_{i=0}^{N-1} \varphi(|x_j - x_i|) = \sum_{i=0}^{N-1} w_i \varphi(|x_j - x_i|) \quad (3.7.4)$$

- There is no evidence that either the NRBF method is consistently superior to the RBF method, or vice versa.

Problems in Application

- RBF Interpolation is $O(N^3)$ to initially digest a set of data points, followed by $O(N)$ operations for each interpolated value.
- These workloads effectively limit the usefulness of these general methods to values of $N < 10^4$

Cheating the Limits

- Using Finite Element Methods
- If n is not too large (meaning, usually, $n = 2$), and if the data points are fairly dense, then consider triangulation. Triangulation is an example of a finite element method. Such methods construct some semblance of geometric regularity and then exploit that construction to advantage.

Cheating contd.

- If your accuracy goals will tolerate it, consider moving each data point to the nearest point on a regular Cartesian grid and then using Laplace interpolation (3.8) to fill in the rest of the grid points. After that, you can interpolate on the grid by the methods of 3.6. You will need to compromise between making the grid very fine (to minimize the error introduced when you move the points) and the compute time workload of the Laplace method.

Examples

2-D Examples

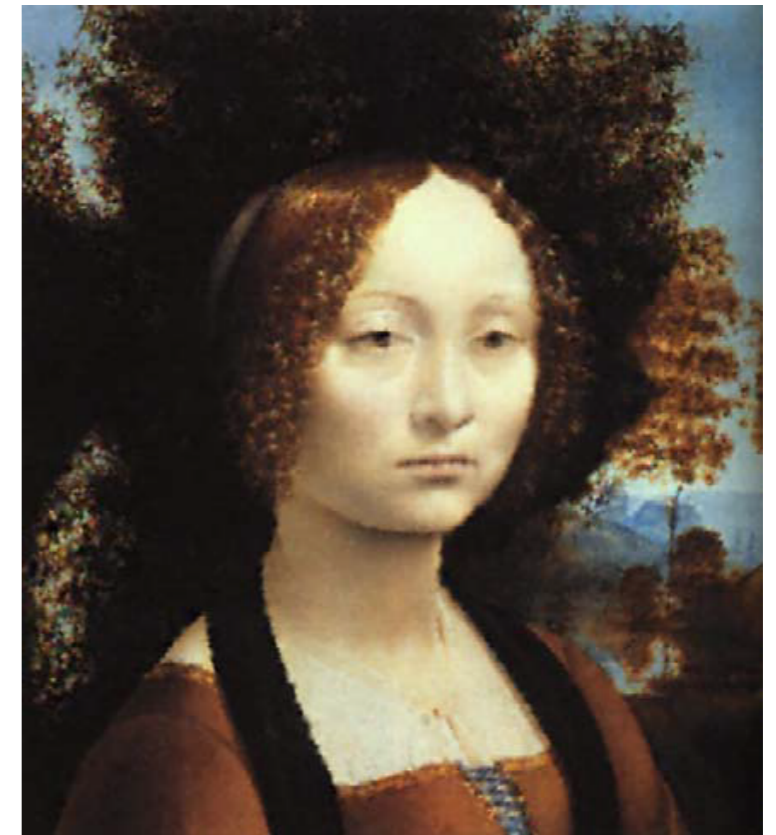
Original



86 % data loss



RBF Interpolation

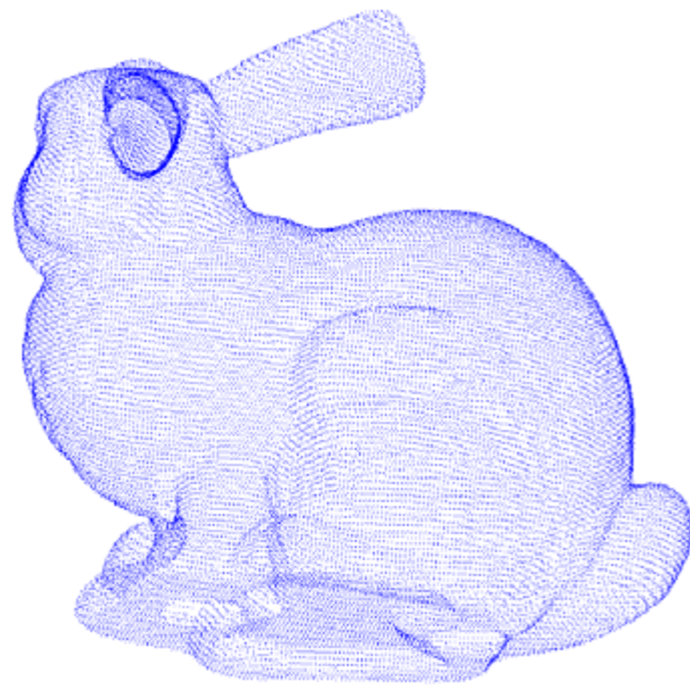


Real-Life Example

- RBFs in 3 dimensions are conceptually similar to fitting a rubber membrane through the measured sample values while minimizing the total curvature of the surface.

3-D Examples

Original unstructured data

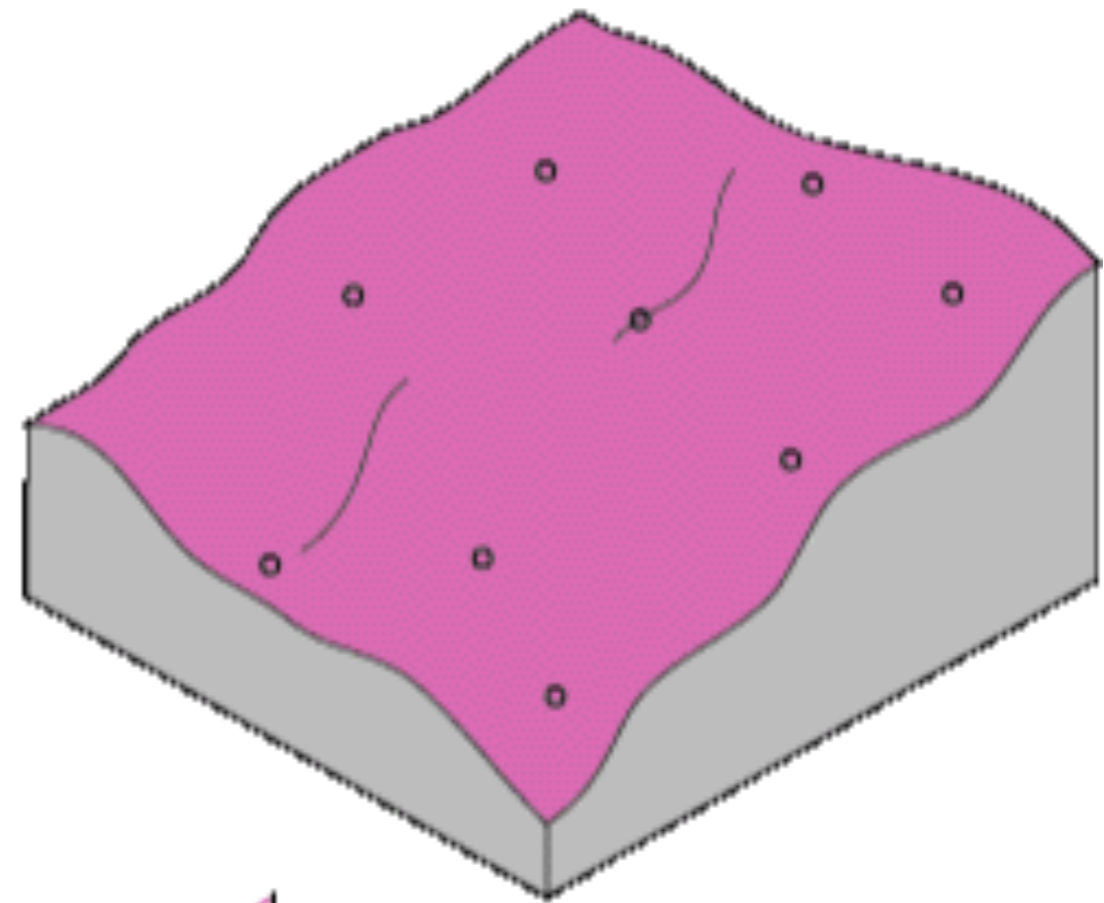
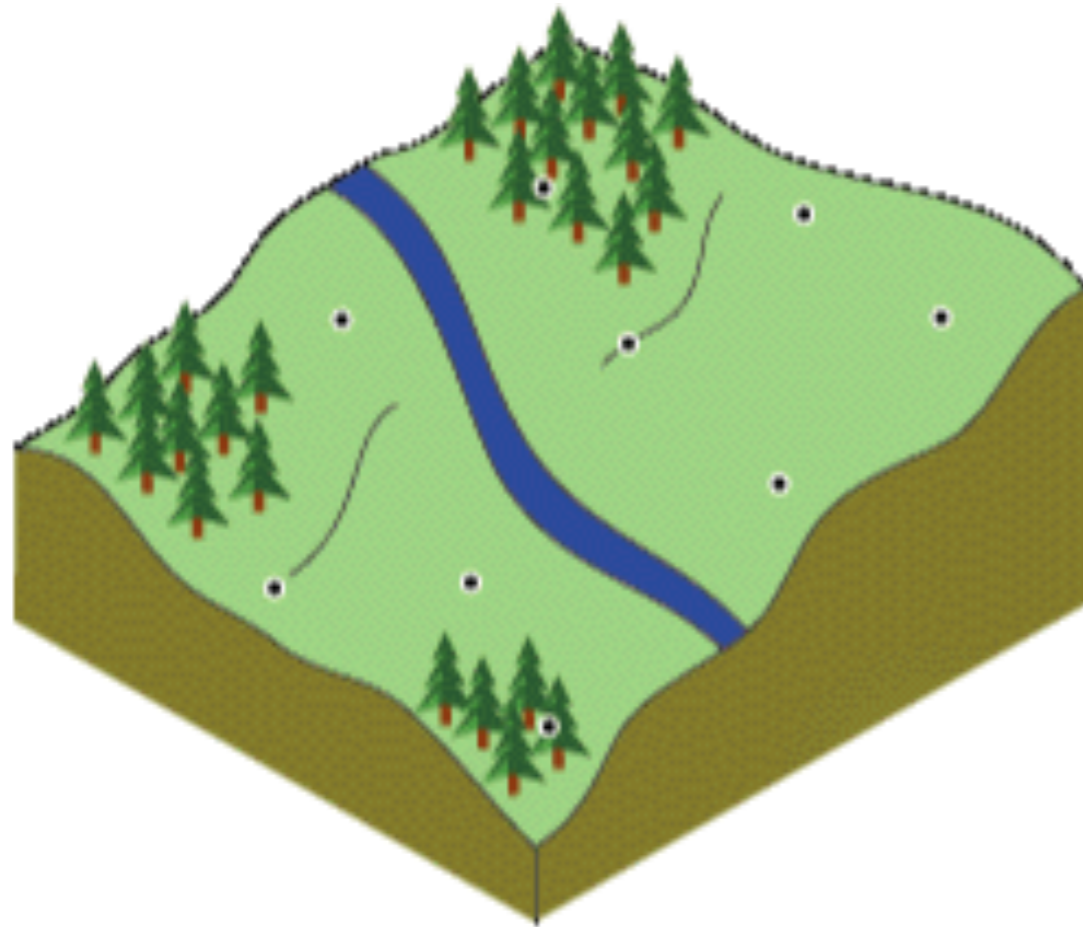


RBF interpolation
to regular spatial grid



3-D Fitting

The diagram below demonstrates conceptually how an RBF surface fits through a series of elevation sample values. Notice in the cross section that the surface passes through the data values.



Sources

- Numerical Recipes Textbook
- http://en.wikipedia.org/wiki/Radial_basis_function
- http://en.wikipedia.org/wiki/Hierarchical_RBF
- [http://webhelp.esri.com/arcgisdesktop/9.2/index.cfm?TopicName=How_Radial_Basis_Functions_\(RBF\)_work](http://webhelp.esri.com/arcgisdesktop/9.2/index.cfm?TopicName=How_Radial_Basis_Functions_(RBF)_work)
- <http://www.umiacs.umd.edu/~ramani/pubs/GumerovDuraishwamiterativeRBF06.pdf>