Cholesky Decomposition

Stephen D. Butalla & Valerie Kobzarenko

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Cholesky Decomposition

- Used when the matrix in question is symmetric and positive definite

  Symmetric matrix: Let $\mathbb{A}$ be a symmetric $n \times n$ matrix. $\mathbb{A}$ is symmetric if

  $$\mathbb{A} = \mathbb{A}^T$$

  i.e., if $a_{ij} = a_{ji}$.
  Example:

  $$\mathbb{A} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & 6 \\ 2 & 6 & -4 \end{pmatrix}$$

  Positive definite:

  1. Let $\mathbf{v}$ be a $n \times 1$ column vector with $v_{ij} \in \mathbb{R}$. $\mathbb{A}$ is positive definite if

     $$\mathbf{v}^T \mathbb{A} \mathbf{v} > 0.$$

  2. $\mathbb{A}$ is positive definite $\iff$ the eigenvalues are positive.

Advantages of Cholesky Decomposition

- $\sim$2 times faster than other methods of solving linear equations
- Numerically stable without pivoting
- Also works for positive-definite Hermitian matrices:

  Hermitian Matrix: Let $\mathbb{A}$ be a square, complex-valued matrix. $\mathbb{A}$ is Hermitian if

  $$\mathbb{A}^{\dagger} = \mathbb{A}$$

  where the dagger ($\dagger$) represents the complex conjugate transpose operation
- Estimated efficiency is on the order of $\mathcal{O}(n^3)$ [1, 2]

Cholesky Decomposition Algorithm

- Cholesky decomposition takes the positive-definite and Hermitian coefficient matrix and breaks it into two matrices:

$$\mathbb{A} = \mathbb{L} \cdot \mathbb{L}^{\dagger}$$

  where $\mathbb{L}$ is a lower triangular matrix and $\mathbb{L}^{\dagger}$ is an upper triangular matrix:

$$\begin{aligned} \mathbb{A} &= \mathbb{L} \cdot \mathbb{L}^{\dagger} \\ &= \begin{pmatrix} l_{00} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ l_{n0} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} l^{\dagger}_{00} & \cdots & l^{\dagger}_{0n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l^{\dagger}_{nn} \end{pmatrix} \end{aligned}$$

- $\mathbb{L}$ and $\mathbb{L}^{\dagger}$ are then found by a recursive expression

Cholesky Decomposition Algorithm

- Diagonal elements are found by [1]:

$$l_{jj} = \sqrt{a_{jj} - \sum_{k=0}^{j} \left[l_{jk} \cdot \left(l_{jk}^{*} \right)^{T} \right]}$$

Example: The first element we find is l_{00}

$$l_{00} = \sqrt{a_{00}}$$

- Off-diagonal elements are found by [1]:

$$l_{ij} = a_{ij} - \sum_{k=0}^{j} \left[l_{ik} \cdot \left(l_{jk}^{*} \right)^{T} \right] \left(a_{jj} - \sum_{k=0}^{j} \left[l_{jk} \cdot \left(l_{jk}^{*} \right)^{T} \right] \right)^{-1/2}$$

- Decomposition into $\mathbb{L}$ and $\mathbb{L}^{\dagger}$ is recursive $\Rightarrow$ the next element depends on the previous element; we solve in a pattern shown in Fig. 1

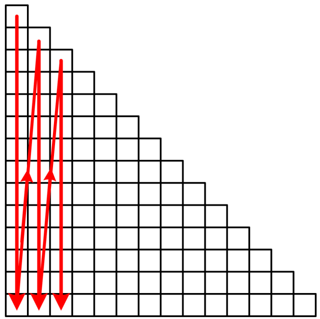

Fig. 1: Order of finding the matrix elements for $\mathbb{L}$

Code Example

Adapted from [2]:

```
//Check if matrix is positive-definite
//and compute decomposition
for (i = 0; i < n; i++) {

        for ( j = i; j < n; j++) {
                //Coefficient matrix previously stored in el[][]
                for ( sum = el[i][j], k= i-1; k >= 0; k--){

                        sum -= el[i][k]*el[j][k];
                }
                //Compute diagonal elements
                if (i == j) {
                        if (sum <= 0.0){
                                throw("Cholesky failed");
                        }

                        el[i][i]=sqrt(sum);
                }
                //Compute off-diagonal elements
                else {

                        el[j][i]=sum/el[i][i];
                }
        }       //Upper triangular elements assigned to zero
                for (i=0;i<n;i++) for (j=0;j<i;j++) el[j][i] = 0.;
}
```

Cholesky Decomposition Algorithm

- Now solve the original equation $\mathbb{A} \cdot \mathbf{x} = \mathbf{b}$ by substituting:

$$\mathbb{L} \cdot \mathbb{L}^{\dagger} \cdot \mathbf{x} = \mathbf{b}$$

- Define [1]

$$\mathbf{y} \equiv \mathbb{L}^{\dagger} \cdot \mathbf{x} \;\Rightarrow\; \mathbb{L} \cdot \mathbf{y} = \mathbf{b}$$

and use forward substitution to find $\mathbf{y}$; i.e.,

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} y_j \cdot l_{ij}}{l_{ii}}$$

from $i = 1 \rightarrow n$ (recall that $\mathbb{L}$ is a bottom triangular matrix)

```
//Forward substitution to find y_{i}
for (i=0;i<n;i++){

        for (sum=b[i],j=0; j<i; j++){

                sum -= el[i][j]*y[j];
        }
        y[i] = sum/el[i][i];
}
```

Cholesky Decomposition Algorithm

- Finally, use back substitution to solve

$$\mathbb{L}^{\dagger} \cdot \mathbf{x} = \mathbf{y}$$

for x_i:

$$x_i = \frac{y_i - \sum_{j=i+1}^{n} y_i \cdot l_{ij}^{*}}{l_{ii}}$$

```
//Back substitution to find x_{i}
for (i = n - 1; i >= 0; i--){

        for (sum=x[i], k = i+1; k < n; k++){

                sum -= el[k][i]*x[k];
        }
        x[i] = sum / el[i][i];
}
```

Cholesky Decomposition Example

$$\mathbb{A} \cdot \mathbf{x} = \mathbf{b}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

Verifying this matrix is symmetric (and Hermitian):

$$\mathbb{A} = \mathbb{A}^{\dagger} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and positive-definite $\Rightarrow$ solve the ev problem:

$$\det\left(\mathbb{A} - \lambda \mathbb{I}\right) = 0$$

$$\Rightarrow \ \lambda_1 = 2, \ \lambda_2 = 2 + \sqrt{2}, \ \lambda_3 = 2 - \sqrt{2}$$

$\because \lambda_1, \lambda_2, \lambda_3 > 0$, this matrix is positive-definite.

Cholesky Decomposition Example

Finding the elements of $\mathbb{L}$:

$$l_{jj} = \sqrt{a_{jj} - \sum_{k=0}^{j} \left[l_{jk} \cdot \left(l_{jk}^{*} \right)^{T} \right]}$$

$$l_{ij} = a_{ij} - \sum_{k=0}^{j} \left[l_{ik} \cdot \left(l_{jk}^{*} \right)^{T} \right] \left(a_{jj} - \sum_{k=0}^{j} \left[l_{jk} \cdot \left(l_{jk}^{*} \right)^{T} \right] \right)^{-1/2}$$

$$l_{00} = \sqrt{a_{00}} = \sqrt{2},\ \ l_{10} = \frac{a_{10}}{l_{00}} = -\frac{1}{\sqrt{2}},\ \ l_{20} = \frac{a_{20}}{l_{00}} = 0$$

$$l_{11} = \sqrt{a_{11} - \left(l_{10} l_{10}^{*} \right)} = \sqrt{2 - (1/2)} = \sqrt{3/2},\ \ l_{21} = \frac{a_{21} - l_{20} l_{10}^{*}}{l_{11}} = \frac{-1}{\sqrt{3/2}} = -\sqrt{\frac{2}{3}}$$

$$l_{22} = \sqrt{a_{22} - \left[l_{20} l_{20}^{*} + l_{21} l_{21}^{*} \right]} = \frac{2}{\sqrt{3}}$$

Cholesky Decomposition Example

$$\mathbb{L} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -2^{-1/2} & (3/2)^{1/2} & 0 \\ 0 & -(2/3)^{1/2} & 2(3^{-1/2}) \end{pmatrix}$$

It follows that $\mathbb{L}^\dagger$ is:

$$\mathbb{L}^\dagger = \begin{pmatrix} \sqrt{2} & -2^{-1/2} & 0 \\ 0 & (3/2)^{1/2} & -(2/3)^{1/2} \\ 0 & 0 & 2(3^{-1/2}) \end{pmatrix}$$

Now, solve $\mathbb{L} \cdot \mathbf{y} = \mathbf{b}$ using forward substitution:

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ -2^{-1/2} & (3/2)^{1/2} & 0 \\ 0 & -(2/3)^{1/2} & 2(3^{-1/2}) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$

$$y_0 = \frac{4}{\sqrt{2}}$$

$$y_1 = \frac{2 + 2^{-1/2} y_0}{(3/2)^{1/2}} = 4\sqrt{\frac{3}{2}}$$

$$y_2 = \sqrt{2}\left(6 + \sqrt{\frac{3}{2}} y_1\right) = 5\sqrt{3}$$

Cholesky Decomposition Example

Now, solve $\mathbb{L}^\dagger \cdot \mathbf{x} = \mathbf{y}$ using back substitution:

$$\begin{pmatrix} \sqrt{2} & -2^{-1/2} & 0 \\ 0 & (3/2)^{1/2} & -(2/3)^{1/2} \\ 0 & 0 & 2(3^{-1/2}) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4/\sqrt{2} \\ 4\sqrt{\frac{3}{2}} \\ 5\sqrt{3} \end{pmatrix}$$

$$\begin{aligned} x_2 &= 6.5 \\ x_1 &= 7 \\ x_0 &= 5.5 \end{aligned}$$

Verifying this solution is correct:

$$\begin{aligned} 2x_0 - x_1 &= 11 - 7 = 4 && \Rightarrow\ 4 = 4 \\ -x_0 + 2x_1 - x_2 &= -5.5 + 14 - 6.5 && \Rightarrow\ 2 = 2 \\ -x_1 + 2x_2 &= -7 + 13 && \Rightarrow\ 6 = 6 \end{aligned}$$

References

[1] M. Parker, *Digital Signal Processing 101: Everything You Need to Know to Get Started*, 2017, Elsevier Inc.

[2] W. Press, S. Teukolsky, W. Vetterling, & B. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 2007, Cambridge University Press.