LUCAS TRAMASSO

LU DECOMPOSITION

1. DEFINITION

- Used to solve systems of linear equations.
- Important when inverting a matrix or calculating a determinant.
- Breaks the linear set into two trivial sets of equations.

1. DEFINITION

We want to find a solution for:

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

1. DEFINITION

Suppose we could write A as:

$$A = L \cdot U$$

Where L is Lower triangular and U is Upper triangular

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{00} & 0 & 0 \\ l_{10} & l_{11} & 0 \\ l_{20} & l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{00} & u_{01} & u_{02} \\ 0 & u_{11} & u_{12} \\ 0 & 0 & u_{22} \end{bmatrix}$$

1. DEFINITION

From our initial system

$$Ax = b$$

Using the matrices L and U we can write the linear set as:

$(L \cdot U)x = b$, or also $L(U \cdot x) = b$

1. DEFINITION

We can now solve the linear set

$$L(U \cdot x) = b$$

by solving it in two separate parts

$L \cdot y = b$ and then $U \cdot x = y$

Forward Substitution

Backsubstitution

2. METHODS

- Pivoting is important when the ordering in the matrix makes the decomposition fail.
- Proper permutation may solve the issue.
- $PA = LU$ where P is a permutation matrix, that reorders A, also called LUP Decomposition (Partial Pivoting).
- Crout's Method, avoids using Gaussian Elimination.
- Other Methods.

3. EXAMPLE USING GAUSSIAN ELIMINATION

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

3. EXAMPLE USING GAUSSIAN ELIMINATION

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$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad R2 = -1R1 + R2 \quad L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ & & 1 \end{bmatrix}$$

3. EXAMPLE USING GAUSSIAN ELIMINATION

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 2 & 3 & 1 \end{bmatrix} \quad R3 = -2R1 + R3 \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ & & 1 \end{bmatrix}$$

3. EXAMPLE USING GAUSSIAN ELIMINATION

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & & 1 \end{bmatrix}$$

3. EXAMPLE USING GAUSSIAN ELIMINATION

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \qquad R3 = \frac{1}{3}R2 + R3 \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & & 1 \end{bmatrix}$$

3. EXAMPLE USING GAUSSIAN ELIMINATION

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix}$$

3. EXAMPLE USING GAUSSIAN ELIMINATION

Now solve:

$L \cdot y = b$ and $U \cdot x = y$

$$\mathrm{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

- $y_1 = 4$
- $y_1 + y_2 = -6$
- $2y_1 - \frac{y_2}{3} + y + 3 = 7$

$y_2 = -10$

$y_3 = -\frac{13}{3}$

3. EXAMPLE USING GAUSSIAN ELIMINATION

Now solve:

$L \cdot y = b$ and $U \cdot x = y$

- $x_1 + x_2 - x_3 = 4$
- $-3x_2 + 4x_3 = -10$
- $\frac{13}{3}x_3 = -\frac{13}{3}$

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 4 \\ 0 & 0 & \frac{13}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -\frac{13}{3} \end{bmatrix}$$

$$\boxed{x_1 = 1 \qquad x_2 = 2 \qquad x_3 = -1}$$

4. EXAMPLE USING CROUT'S METHOD

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

Find A = LU, in this case the diagonal of U should have 1's

$$L = \begin{bmatrix} l_{00} & 0 & 0 \\ l_{10} & l_{11} & 0 \\ l_{20} & l_{21} & l_{22} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{01} & u_{02} \\ 0 & 1 & u_{12} \\ 0 & 0 & 1 \end{bmatrix} \quad LU = \begin{bmatrix} l_{00} & l_{00}u_{01} & l_{00}u_{02} \\ l_{10} & l_{10}u_{01} + l_{11} & l_{10}u_{02} + l_{11}u_{12} \\ l_{20} & l_{20}u_{01} + l_{21} & l_{20}u_{02} + l_{21}u_{12} + l_{22} \end{bmatrix}$$

4. EXAMPLE USING CROUT'S METHOD

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

Now compute A = LU

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} = \begin{bmatrix} l_{00} & l_{00}u_{01} & l_{00}u_{02} \\ l_{10} & l_{10}u_{01} + l_{11} & l_{10}u_{02} + l_{11}u_{12} \\ l_{20} & l_{20}u_{01} + l_{21} & l_{20}u_{02} + l_{21}u_{12} + l_{22} \end{bmatrix}$$

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$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

We now have

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 14 & 0 \\ -2 & 2 & -\frac{3}{7} \end{bmatrix} \quad U = \begin{bmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

4. EXAMPLE USING CROUT'S METHOD

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

Solve $L \cdot y = b$

$$\begin{bmatrix} 2 & 0 & 0 \\ 6 & 14 & 0 \\ -2 & 2 & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

$$y_1 = 2$$

$$y_2 = -\frac{1}{7}$$

$$y_3 = 4$$

4. EXAMPLE USING CROUT'S METHOD

$$\begin{bmatrix} 2 & -4 & 1 \\ 6 & 2 & -1 \\ -2 & 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -6 \end{bmatrix}$$

Solve $U \cdot x = y$

$$\begin{bmatrix} 1 & -2 & \frac{1}{2} \\ 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{7} \\ 4 \end{bmatrix}$$

$$x_1 = 2$$

$$x_2 = 1$$

$$x_3 = 4$$