Barycentric Interpolation and Coefficients

Mahbuba Perveen

10/20/2020

Rational Functions

A rational function is a ratio of polynomials **$p(x)/q(x)$**.

If the numerator **$p(x)$** and the denominator **$q(x)$** have no roots in common, then the rational function is in **reduced form**.

$(x^2+1)/(x^3+3x+1)$ is in reduced form.

$(x-2)/(x^2-4)$ is not in reduced form, because **$x=2$** is a root of both numerator and denominator.

$(x-2)/(x^2-4) = (x-2)/(x-2)(x+2) = 1/(x+2)$

Poles

For a rational function in reduced form, the poles are the values of x where the denominator is equal to zero.

In other words, the rational function is not defined at its poles.

Example:

The function $1/(x^2+8x+7)$ has poles at $x=-1$ and $x=-7$

The function $(x-2)/(x^2-4) = 1/(x+2)$ has only one pole, $x=-2$

The function (x^2+1) has no poles

Poles (cont.)

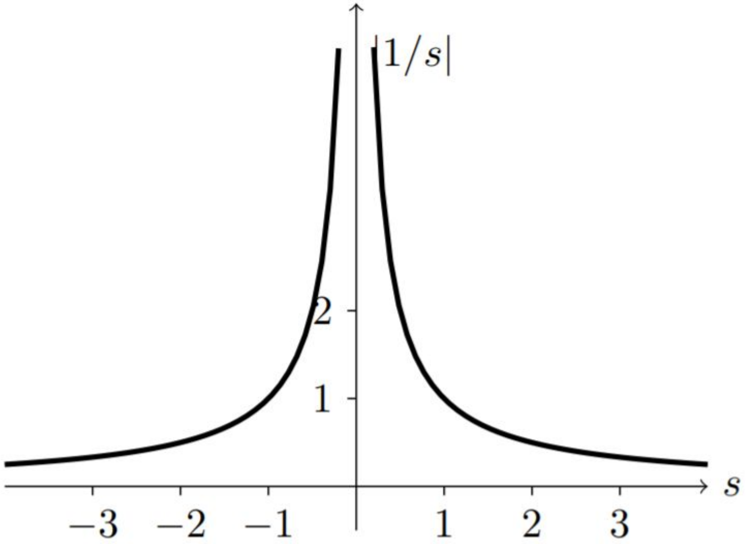

Figure 1: Graph of $\frac{1}{|s|}$ for s real.

Rational Interpolation

$$R_{i(i+1)\dots(i+m)} = \frac{P_\mu(x)}{Q_\nu(x)} = \frac{p_0 + p_1 x + \cdots + p_\mu x^\mu}{q_0 + q_1 x + \cdots + q_\nu x^\nu} \qquad (3.4.1)$$

$$m + 1 = \mu + \nu + 1 \qquad (3.4.2)$$

For the interpolation problem, a rational function is constructed to go through a set of tabulated functional values.

While constructing a global approximation on the entire table of values using all the given nodes x_0, x_1, … x_{N-1}, one potential drawback is that the approximation can have poles inside the interpolation interval, even if the original function has no poles there.

Rational Interpolation (cont.)

- We can make the degree of both the numerator and the denominator in eqn. (3.4.1) be N-1
  - There would be no poles anywhere on the real axis
  - Allows the actual order of approximation to be specified to be any integer **d<N**
- This requires that the p's and the q's not be independent, so that eqn. 3.4.2 no longer holds

Barycentric Algorithm

Barycentric form of the rational interpolant:

$$R(x) = \frac{\displaystyle\sum_{i=0}^{N-1} \frac{w_i}{x - x_i} y_i}{\displaystyle\sum_{i=0}^{N-1} \frac{w_i}{x - x_i}}$$

$$w_k = \sum_{\substack{i=k-d \\ 0 \le i < N-d}}^{k} (-1)^k \prod_{\substack{j=i \\ j \neq k}}^{i+d} \frac{1}{x_k - x_j}$$

Example:

$$\begin{aligned} w_k &= (-1)^k, & d &= 0 \\ w_k &= (-1)^{k-1} \left[\frac{1}{x_k - x_{k-1}} + \frac{1}{x_{k+1} - x_k} \right], & d &= 1 \end{aligned}$$

N is the number of nodes, d is the desired order

Barycentric Interpolation

- By a suitable choice of the weights $\mathbf{w_i}$, every rational interpolant can be written in the barycentric form.
  - As a special case, polynomial interpolants as well
- Barycentric rational interpolation competes very favorably with splines
  - It's error is often smaller
  - The resulting approximation is infinitely smooth (unlike splines)
- If we want our rational interpolant to have approximation order **d**, i.e., if the spacing of the points is $O(h)$, the error is $O(h^{d+1})$ as $h \to 0$

Runge's example with Barycentric Interpolation

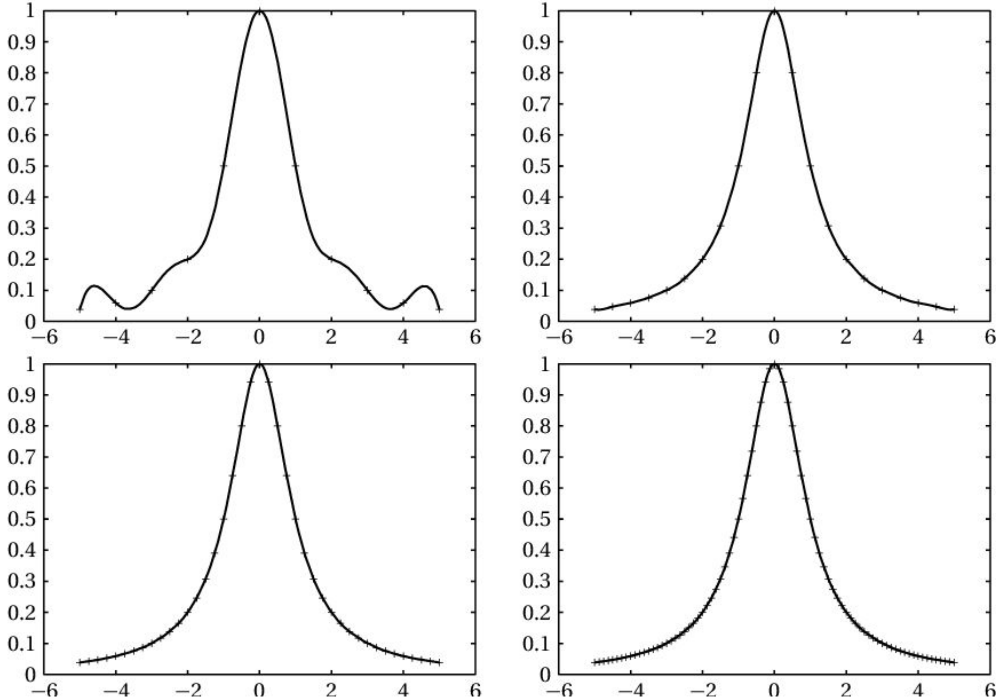

Figure: Interpolating Runge's example with d = 3 and n = 10, 20, 40, 80.

Coefficients of Polynomials

Coefficients of the Interpolating Polynomial

- Sometimes we may need the coefficients of a polynomial, rather than the actual value of the interpolating polynomial
    - For example, to compute simultaneous interpolated values of the function and several of its derivatives
    - To convolve a segment of the tabulated function with some other function, where the moments of the other function (i.e., its convolution with powers of x) are known analytically
- Generally the coefficients of the interpolating polynomial can be determined much less accurately than its value at a desired abscissa
    - Therefore, it is not a good idea to determine the coefficients only for use in calculating interpolating values
    - Interpolated values calculated this way will not pass exactly through the tabulated points

Vandermonde Matrix

Let's take the tabulated points to be: $y_i \equiv y(x_i)$

If the interpolating polynomial is written as: $y = c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1} x^{N-1}$

Then the c_i's are required to satisfy the linear equation:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}$$

This is a Vandermonde matrix.

Coefficients (cont.)

- For high degrees of interpolation, precision of coefficients are essential
  - Interpolation error is compounded by inaccuracy of coefficients
- Vandermonde systems are notoriously ill-conditioned
  - In such cases, no numerical method gives a very accurate result
- Only practical for small datasets
  - As N increases, the Vandermonde system becomes more ill-conditioned
- It’s better to compute Vandermonde problems in double precision or higher

References

[1]https://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-iii-fourier-series-and-laplace-transform/poles-amplitude-response-connection-to-erf/MIT18_03SCF11_s31_1text.pdf

[2]Press, William H., and William T. Vetterling. Numerical Recipes. Cambridge Univ. Press, 2007.

[3]https://www.semanticscholar.org/paper/Barycentric-rational-interpolation-with-no-poles-of-Floater-Hormann/221ed06a9edf2f0f2c96dd062d20994d6eb07abb/figure/0