Barycentric Interpolation and Coefficients

Mahbuba Perveen
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Rational Functions

A rational function is a ratio of polynomials $p(x)/q(x)$.

If the numerator $p(x)$ and the denominator $q(x)$ have no roots in common, then the rational function is in reduced form.

$\frac{x^2+1}{x^3+3x+1}$ is in reduced form.

$\frac{x-2}{x^2-4}$ is not in reduced form, because $x=2$ is a root of both numerator and denominator.

$\frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2}$
Poles

For a rational function in reduced form, the poles are the values of $x$ where the denominator is equal to zero.

In other words, the rational function is not defined at its poles.

Example:

The function $\frac{1}{x^2+8x+7}$ has poles at $x=-1$ and $x=-7$

The function $\frac{x-2}{x^2-4} = \frac{1}{x+2}$ has only one pole, $x=-2$

The function $(x^2+1)$ has no poles
Poles (cont.)

Figure 1: Graph of $\frac{1}{|s|}$ for $s$ real.
Rational Interpolation

\[ R_{i(i+1)\ldots(i+m)} = \frac{P_\mu(x)}{Q_v(x)} = \frac{p_0 + p_1x + \cdots + p_\mu x^\mu}{q_0 + q_1x + \cdots + q_v x^v} \quad (3.4.1) \]

\[ m + 1 = \mu + v + 1 \quad (3.4.2) \]

For the interpolation problem, a rational function is constructed to go through a set of tabulated functional values.

While constructing a global approximation on the entire table of values using all the given nodes \( x_0, x_1, \ldots x_{N-1} \), one potential drawback is that the approximation can have poles inside the interpolation interval, even if the original function has no poles there.
Rational Interpolation (cont.)

- We can make the degree of both the numerator and the denominator in eqn. (3.4.1) be N-1
  - There would be no poles anywhere on the real axis
  - Allows the actual order of approximation to be specified to be any integer $d < N$
- This requires that the $p$'s and the $q$'s not be independent, so that eqn. 3.4.2 no longer holds
Barycentric Algorithm

Barycentric form of the rational interpolant:

\[ R(x) = \frac{\sum_{i=0}^{N-1} w_i y_i}{\sum_{i=0}^{N-1} w_i} \]

\[ w_k = \sum_{i=k-d}^{0 \leq i < N-d} (-1)^k \prod_{j=i}^{j \neq k} \frac{1}{x_k - x_j} \]

Example:

\[ w_k = (-1)^k, \quad d = 0 \]

\[ w_k = (-1)^{k-1} \left[ \frac{1}{x_k - x_{k-1}} + \frac{1}{x_{k+1} - x_k} \right], \quad d = 1 \]

N is the number of nodes, d is the desired order
Barycentric Interpolation

- By a suitable choice of the weights $w_i$, every rational interpolant can be written in the barycentric form.
  - As a special case, polynomial interpolants as well
- Barycentric rational interpolation competes very favorably with splines
  - It’s error is often smaller
  - The resulting approximation is infinitely smooth (unlike splines)
- If we want our rational interpolant to have approximation order $d$, i.e., if the spacing of the points is $O(h)$, the error is $O(h^{d+1})$ as $h \to 0$
Runge’s example with Barycentric Interpolation

Figure: Interpolating Runge's example with $d = 3$ and $n = 10, 20, 40, 80$. 
Coefficients of Polynomials
Coefficients of the Interpolating Polynomial

- Sometimes we may need the coefficients of a polynomial, rather than the actual value of the interpolating polynomial
  - For example, to compute simultaneous interpolated values of the function and several of its derivatives
  - To convolve a segment of the tabulated function with some other function, where the moments of the other function (i.e., its convolution with powers of x) are known analytically

- Generally the coefficients of the interpolating polynomial can be determined much less accurately than its value at a desired abscissa
  - Therefore, it is not a good idea to determine the coefficients only for use in calculating interpolating values
  - Interpolated values calculated this way will not pass exactly through the tabulated points
Vandermonde Matrix

Let’s take the tabulated points to be: $y_i \equiv y(x_i)$

If the interpolating polynomial is written as: $y = c_0 + c_1 x + c_2 x^2 + \cdots + c_{N-1} x^{N-1}$

Then the $c_i$’s are required to satisfy the linear equation:

$$
\begin{bmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^{N-1} \\
1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{N-1} & x_{N-1}^2 & \cdots & x_{N-1}^{N-1}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{N-1}
\end{bmatrix}
=
\begin{bmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{N-1}
\end{bmatrix}
$$

This is a Vandermonde matrix.
Coefficients (cont.)

- For high degrees of interpolation, precision of coefficients are essential
  - Interpolation error is compounded by inaccuracy of coefficients
- Vandermonde systems are notoriously ill-conditioned
  - In such cases, no numerical method gives a very accurate result
- Only practical for small datasets
  - As N increases, the Vandermonde system becomes more ill-conditioned
- It’s better to compute Vandermonde problems in double precision or higher
References


[3] https://www.semanticscholar.org/paper/Barycentric-rational-interpolation-with-no-poles-of-Floater-Hormann/221ed06a9edf2f0f2c96dd062d20994d6eb07abb/figure/e0