

Singular Value Decomposition (SVD) A Fast Track Tutorial

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First Published on September 11, 2006; Last Update: September 12, 2006
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Abstract

This fast track tutorial provides instructions for decomposing a matrix using the singular value decomposition (SVD) algorithm. The tutorial covers singular values, right and left eigenvectors and a shortcut for computing the full SVD of a matrix.

Keywords

singular value decomposition, SVD, singular values, eigenvectors, full SVD, matrix decomposition

Problem: Compute the full SVD for the following matrix:

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Solution:

Step 1. Compute its transpose \mathbf{A}^T and $\mathbf{A}^T\mathbf{A}$.

Since $\mathbf{A}^T = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix}$ then, $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

Step 2. Determine the eigenvalues of $\mathbf{A}^T\mathbf{A}$ and sort these in descending order, in the absolute sense. Square roots these to obtain the singular values of \mathbf{A} .

$$\mathbf{A}^T\mathbf{A} - c\mathbf{I} = \begin{bmatrix} 25 - c & -15 \\ -15 & 25 - c \end{bmatrix} \quad |\mathbf{A}^T\mathbf{A} - c\mathbf{I}| = (25 - c)(25 - c) - (-15)(-15) = 0$$

characteristic equation $\longrightarrow c^2 - 50c + 400 = 0$

The quadratic equation gives two values.
In decreasing order, these are $\longrightarrow |40| > |10|$

eigenvalues $\longrightarrow c_1 = 40 \quad c_2 = 10$

singular values $\longrightarrow s_1 = \sqrt{40} = 6.3245 > s_2 = \sqrt{10} = 3.1622$

Step 3. Construct diagonal matrix \mathbf{S} by placing singular values in descending order along its diagonal. Compute its inverse, \mathbf{S}^{-1} .

$$\mathbf{S} = \begin{bmatrix} 6.3245 & 0 \\ 0 & 3.1622 \end{bmatrix} \quad \mathbf{S}^{-1} = \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix}$$

Step 4. Use the ordered eigenvalues from step 2 and compute the eigenvectors of $\mathbf{A}^T\mathbf{A}$. Place these eigenvectors along the columns of \mathbf{V} and compute its transpose, \mathbf{V}^T .

for $c_1 = 40$

$$\mathbf{A}^T\mathbf{A} - c\mathbf{I} = \begin{bmatrix} 25 - 40 & -15 \\ -15 & 25 - 40 \end{bmatrix} = \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A} - c\mathbf{I}) \mathbf{x}_1 = \mathbf{0}$$

$$\begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-15x_1 + -15x_2 = 0$$

$$-15x_1 + -15x_2 = 0$$

Solving for x_2 for either equation: $x_2 = -x_1$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$$

Dividing by its length,

$$L = \sqrt{x_1^2 + x_2^2} = x_1\sqrt{2}$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 / L \\ -x_1 / L \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$$

for $c_2 = 10$

$$\mathbf{A}^T\mathbf{A} - c\mathbf{I} = \begin{bmatrix} 25 - 10 & -15 \\ -15 & 25 - 10 \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A} - c\mathbf{I}) \mathbf{x}_2 = \mathbf{0}$$

$$\begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$15x_1 + -15x_2 = 0$$

$$-15x_1 + 15x_2 = 0$$

Solving for x_2 for either equation: $x_2 = x_1$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

Dividing by its length,

$$L = \sqrt{x_1^2 + x_2^2} = x_1\sqrt{2}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 / L \\ x_1 / L \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$$

$$\mathbf{V}^T = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

Step 5. Compute \mathbf{U} as $\mathbf{U} = \mathbf{AVS}^{-1}$. To complete the proof, compute the full SVD using $\mathbf{A} = \mathbf{USV}^T$.

$$\mathbf{U} = \mathbf{AVS}^{-1} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 0.1581 & 0 \\ 0 & 0.3162 \end{bmatrix}$$

$$\mathbf{U} = \mathbf{AVS}^{-1} = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0.1118 & 0.2236 \\ -0.1118 & 0.2236 \end{bmatrix}$$

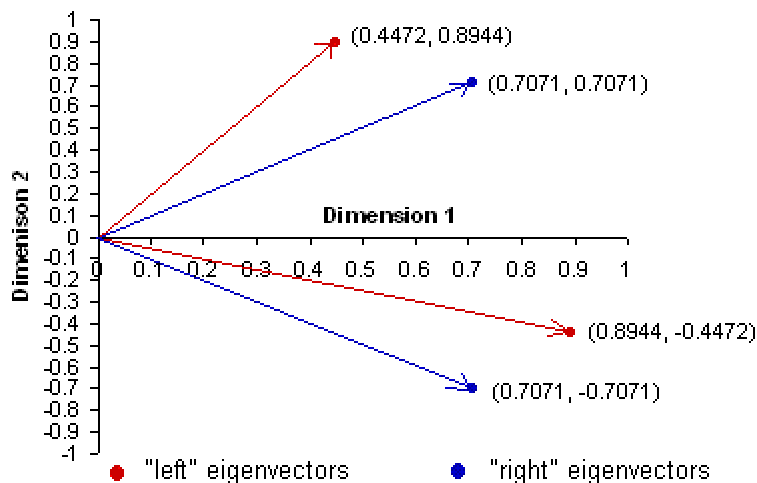
$$\mathbf{U} = \mathbf{AVS}^{-1} = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{USV}^T = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix} \begin{bmatrix} 6.3245 & 0 \\ 0 & 3.1622 \end{bmatrix} \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{USV}^T = \begin{bmatrix} 0.4472 & 0.8944 \\ 0.8944 & -0.4472 \end{bmatrix} \begin{bmatrix} 4.4721 & -4.4721 \\ 2.2360 & 2.2360 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{USV}^T = \begin{bmatrix} 3.9998 & 0 \\ 2.9999 & -4.9997 \end{bmatrix} \approx \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

The orthogonal nature of the \mathbf{V} and \mathbf{U} matrices is evident by inspecting their eigenvectors. This can be demonstrated by computing dot products between column vectors. All dot products are equal to zero. Alternatively, we can plot these and see they are all orthogonal.



Questions

For the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

1. Compute the eigenvalues of $\mathbf{A}^T\mathbf{A}$.
2. Prove that this is a matrix of Rank 2.
3. Compute its full SVD.
4. Compute its Rank 2 Approximation.

References

1. <http://www.miislita.com/information-retrieval-tutorial/svd-lsi-tutorial-3-full-svd.html>
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