HOMEWORK #1

4)

LET $V(T)$ = THE NUMBER OF VERTICES IN BINARY TREE T.

THEOREM. $$V(T) \le 2^{h+1} - 1$$

PROOF. (BY INDUCTION ON h)

BASIS: $h=0$

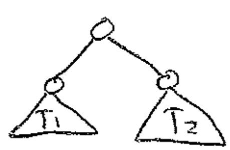
THEN T IS A SINGLE NODE, AND CLEARLY

$$\begin{aligned} V(T) &= 1 \\ &\le 2^{0+1} - 1 \\ &= 1 \quad \checkmark \end{aligned}$$

IND. HYP.:

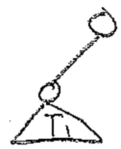
SUPPOSE THERE EXISTS A $K \ge 0$ SUCH THAT $V(T) \le 2^{h+1} - 1$ FOR ALL BINARY TREES WITH $0 \le h \le K$. LET T HAVE HEIGHT $h+1$.

CASE 1) T HAS TWO CHILDREN



$$\begin{aligned} V(T) &= V(T_1) + V(T_2) + 1 \\ &\le (2^{h+1} - 1) + (2^{h+1} - 1) + 1 && \text{BY THE IND. HYP.} \\ &= 2 \cdot 2^{h+1} - 1 \\ &= 2^{(h+1)+1} - 1 \quad \checkmark \end{aligned}$$

CASE 2) T HAS ONE CHILD



$$
\begin{aligned}
v(T) &= v(T_1) + 1 \\
&\leq (2^{h+1} - 1) + 1 \qquad \text{BY THE IND. HYP.} \\
&\leq 2 \cdot (2^{h+1} - 1) + 1 \\
&= 2^{(h+1)+1} - 1
\end{aligned}
$$

$\square$

6) (PROBABLY MAKES A BIT MORE SENSE FOR $n \geq 1$)

THEOREM. $\displaystyle\sum_{i=1}^{n} (3i-1) = \frac{n(3n+1)}{2}$ $\quad n \geq 0$

PROOF. (BY IND. ON n)

BASIS: $n=0$

$$\sum_{i=1}^{0} (3i-1) = 0 = \frac{0(3(0)+1)}{2} = 0$$

IND. HYP.

SUPPOSE THERE EXISTS A $K \geq 0$ SUCH THAT $\displaystyle\sum_{i=1}^{K} (3i-1) = \frac{K(3K+1)}{2}$.

IND. STEP.

$$\sum_{i=1}^{k+1} (3i-1) = \sum_{i=1}^{k} (3i-1) + (3(k+1)-1)$$

$$= \frac{k(3k+1)}{2} + (3(k+1)-1) \quad \text{BY THE IND. HYP.}$$

$$= \frac{k(3k+1)}{2} + \frac{2(3(k+1)-1)}{2}$$

$$\vdots$$

$$= \frac{(k+1)(3(k+1)+1)}{2} \checkmark$$

$\square$

8)

THEOREM.

3 IS A FACTOR OF $n^3 - n + 3$, FOR ALL $n \geq 0$.

PROOF. (BY IND. ON n)

BASIS: $n=0$

$$n^3 - n + 3 = 0^3 - 0 + 3$$
$$= 3$$

CLEARLY, 3 IS A FACTOR OF 3.

IND. HYP.

SUPPOSE THERE EXISTS A $k \geq 0$ SUCH THAT 3 IS A FACTOR OF $k^3 - k + 3$

IND. STEP:

WE WILL SHOW THAT 3 IS A FACTOR OF $(k+1)^3 - (k+1) + 3$.

$$(k+1)^3 - (k+1) + 3 = (k+1)\left[(k+1)^2 - 1\right] + 3$$
$$\vdots$$
$$= k^3 + 3k^2 + 2k + 3$$
$$= (k^3 - k + 3) + (3k^2 + 3k)$$
$$= (k^3 - k + 3) + 3(k^2 + k)$$
$$= 3(z) + 3(k^2 + k)$$
$$= 3(z + k^2 + k)$$

BY THE IND. HYP. $k^3 - k + 3 = 3 \cdot z$ FOR SOME z,

$\square$