HOMEWORK # 2

1) SHOW $NEG = \{ x \mid x \geq 1 \text{ AND } x \text{ IS ODD} \}$ IS COUNTABLY INFINITE, i.e., $|N| = |NEG|$.

LET

$$f(x) = 2x + 1$$

THEN f IS:

1) A TOTAL FUNCTION
2) ONE-TO-ONE
3) ONTO

THUS $f: N \to NEG$ IS A BIJECTION AND $|N| = |NEG|$, THUS NEG IS COUNTABLY INFINITE.

11) LET $\Sigma = \{0,1\}$ AND

$$P(\Sigma^*) = \{ \{\}, \{\epsilon\}, \{0\}, \{1\}, \{0,1\}, \{0,1,\epsilon\}, \ldots \}$$

IN OTHERWORDS, $P(\Sigma^*)$ IS THE SET OF ALL SUBSETS OF Σ^*. WE WILL PROVE THAT $P(\Sigma^*)$ IS UNCOUNTABLE.

PROOF. (BY CONTRADICTION)

SUPPOSE $P(\Sigma^*)$ IS COUNTABLE. THEN $P(\Sigma^*)$ IS EITHER FINITE OR COUNTABLY INFINITE. CLEARLY $P(\Sigma^*)$ IS NOT FINITE. HENCE, IT IS COUNTABLY INFINITE AND THERE IS THUS A BIJECTION: $f: N \rightarrow P(\Sigma^*)$

$$f: \quad \begin{array}{cccc} 0 & 1 & 2 & \\ \downarrow & \downarrow & \downarrow & \ldots \\ P_0 & P_1 & P_2 & \end{array}$$

(EVERY SET IN $P(\Sigma^*)$ OCCURS IN THIS LIST SINCE f IS A BIJECTION)

DEFINE A TABLE:

	w_0 ϵ	w_1 0	w_2 1	w_3 00	w_4 01	...
P_0	d_0^0	d_1^0	d_2^0	d_3^0	...	
P_1	d_0^1	d_1^1	d_2^1	d_3^1	...	
P_2		$\vdots$				
$\vdots$						

WHERE

$$d^i_j = \begin{cases} 0 & \text{IF} \quad \omega_j \notin P_i \\ 1 & \text{IF} \quad \omega_j \in P_i \end{cases} \Bigg]$$

NOW DEFINE A SET $D \subseteq \Sigma^*$ AS

$$\omega_i \in D \quad \text{IFF} \quad \omega_i \notin P_i \qquad (*)$$

i.e., D REPRESENTS THE COMPLEMENT OF THE DIAGONAL.

<u>OBSERVATIONS</u>

1) $D \subseteq \Sigma^*$ AND THUS $D \in P(\Sigma^*)$

2) $D = P_j$ FOR SOME $j \geq 0$ SINCE f IS A BIJECTION.

QUESTION: IS $\omega_j \in D$?

$$\omega_i \in D \quad \text{IFF} \quad \omega_i \notin P_i \qquad \text{BY } (*)$$

THUS $\qquad \omega_j \in D \quad \text{IFF} \quad \omega_j \notin P_j$

SINCE $D = P_j \qquad \omega_j \in P_j$ IFF $\omega_j \notin P_j$ A CONTRADICTION.

THUS D IS NOT IN THE TABLE, f IS <u>NOT</u> A BIJECTION, AND HENCE $|N| \neq |P(\Sigma)|$. $P(\Sigma^*)$ IS THEREFORE UNCOUNTABLE.

$\square$

16) LET MF DENOTE THE SET OF ALL MONOTONIC INCREASING FUNCTIONS. WE WILL SHOW THAT MF IS UNCOUNTABLE.

PROOF. (BY CONTRADICTION)

SUPPOSE MF IS COUNTABLE, THEN MF IS EITHER FINITE OR COUNTABLY INFINITE, CLEARLY MF IS NOT FINITE. HENCE IT IS COUNTABLE INFINITE AND THERE IS A BIJECTION: $f: N \to MF$

$$f: \begin{array}{cccc} 0 & 1 & 2 & \\ \downarrow & \downarrow & \downarrow & \cdots \\ f_0 & f_1 & f_2 & \end{array}$$

(EVERY FUNCTION IN MF OCCURS IN THIS LIST SINCE f IS A BIJECTION)

DEFINE A TABLE:

	0	1	2	3 ...
f_0	d_0^0	d_1^0	d_2^0	d_3^0
f_1	d_0^1	d_1^1	d_2^1	d_3^1
f_2		$\vdots$		
$\vdots$				

WHERE $d_j^i = f_i(j)$

NOW DEFINE A FUNCTION $f_s: N \to N$ AS

$$f_s(i) = \begin{cases} f_i(i) + 1 & \text{IF } i = 0 \\ f_i(i) + f_s(i-1) + 1 & \text{IF } i \neq 0 \end{cases} \quad (*)$$

OBSERVATIONS:

1) f_s IS A FUNCTION FROM N TO N

2) $f_s(n) < f_s(n+1)$, FOR ALL $n \geq 0$, SINCE,

$$f_s(i) = f_i(i) + f_s(i-1) + 1$$

THEREFORE, f_s IS A MONOTONIC INCREASING FUNCTION FROM N TO N, i.e., $f_s \in MF$

3) $f_s = f_j$ FOR SOME $j \geq 0$ SINCE f IS A BIJECTION FROM N TO MF.

QUESTION: WHAT IS $f_s(j)$? $\quad f_s(j) = f_j(j) \quad (**)$

THIS FOLLOWS FROM 3) ABOVE. BUT BY DEFINITION OF f_s IN $(*)$, $f_s(j)$ IS ALSO EQUAL TO THE FOLLOWING DEPENDING ON THE VALUE OF j:

CASE 1) $j = 0$

THEN $f_s(j) = f_j(j) + 1$, BUT $f_j(j) \neq f_j(j) + 1$

CASE 2) $j > 0$

THEN $f_s(j) = f_j(j) + f_s(j-1) + 1$, BUT
$f_j(j) \neq f_j(j) + f_s(j-1) + 1$.

IN EITHER CASE WE GET A CONTRADICTION, THUS f_s IS NOT IN THE TABLE, f IS NOT A BIJECTION, $|MF| \neq |N|$ AND MF IS THEREFORE UNCOUNTABLE.

$\square$