Formal Languages and Automata Theory
Homework # 2

Prove each of the following. Exercises to be handed in included 1, 11, and give 16 a try.

1. Prove that the set $\{x \mid x \geq 1 \text{ and } x \text{ is odd}\}$ is countably infinite.

2. Prove that the set $\{x \mid x \leq 0\}$ is countably infinite.

3. Prove that the set of all integers is countably infinite.

4. Let $\Sigma = \{0, 1\}$. Prove that the set Σ^* is countably infinite.

5. Let Σ be any fixed, finite alphabet. Prove that the set Σ^* is countably infinite.

6. Let $\Sigma = \{0, 1\}$. Prove that the set of all *finite* subsets of Σ^* is countable.

7. Prove that the set of all rational numbers is countable. In other words, the set $\{\frac{m}{n} \mid m, n \in \aleph - \{0\}\}$.

8. Prove that the union of two countable sets is countable.

9. Prove that the Cartesian product of two countable sets is countable.

10. Prove that the set of finite-length sequences consisting of elements of a nonempty countable set is countably infinite.

11. Let $\Sigma = \{0, 1\}$. Prove that the set of all subsets of Σ^* is uncountable.

12. Prove that the set of real numbers in the interval $[0, 1]$ is uncountable.

13. Prove that the set of all (total) functions from $\aleph$ to $\aleph$ is uncountable.

14. Prove that the set of all (total) functions from $\aleph$ to $\{0, 1\}$ is uncountable.

15. A (total) function f from $\aleph$ to $\aleph$ is said to be nonrepeating if $f(n) \neq f(n+1)$, for all $n \in \aleph$. Otherwise, f is said to be repeating. Prove that there is an uncountable number of nonrepeating functions. Also prove that there is an uncountable number of repeating functions.

16. A (total) function f from $\aleph$ to $\aleph$ is said to be monotone increasing if $f(n) < f(n+1)$ for all $n \in \aleph$. Prove that there is an uncountable number of monotone increasing functions.