Non-Context-Free Example

Define the language L as:

$$L = \{0^i 1^i 2^j \mid j \geq i\}$$

Theorem. *L is not a context free language.*

Proof. (By contradiction) Suppose that L is a context free language. Let n be the constant from the pumping lemma, and consider the string $z = 0^n 1^n 2^n$. Clearly, $z \in L$. Since $|z| \geq n$, it follows from the pumping lemma that z can be broken up into five parts i.e., $z = uvwxy$, such that $|vx| \geq 1$, $|vwx| \leq n$, and $uv^iwx^iy \in L$, for all $i \geq 0$. Now consider which parts of $z = 0^n 1^n 2^n$ form the substrings v and x. Since $|vwx| \leq n$ it follows that v and x can contain at most two different symbols. Furthermore, if they do contain two different symbols, than those symbols must be consecutive. In other words, v and x cannot consist of 0's and 2's.

Case 1) v and x consist only of 0's.

Then consider the string $z' = uv^2wx^2y$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and v and x consist only of 0's, it follows that z' is of the form $0^m 1^n 2^n$, where $m > n$. Hence, $z' \notin L$, a contradiction. Similarly if v and x consist only of 1's.

Case 2) v and x consist only of 2's.

Then consider the string $z' = uv^0wx^0y = uwy$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and v and x consist only of 2's, it follows that z' is of the form $0^n 1^n 2^m$, where $n > m$. Hence, $z' \notin L$, a contradiction.

Case 3) v and x consist of both 0's and 1's.

Then consider the string $z' = uv^2wx^2y$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and v and x consist of 0's and 1's, it follows that z' contains more 0's and 1's than 2's. Hence, $z' \notin L$, a contradiction.

Case 4) v and x consist of both 1's and 2's.

Then consider the string $z' = uv^0wx^0y = uwy$. By the pumping lemma $z' \in L$. But since $|vx| \geq 1$, and v and x consist of 1's and 2's, it follows that z' contains fewer 2's than 0's. Hence, $z' \notin L$, a contradiction.□