Formal Languages and Automata
DFA Minimization

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Definition
A state $p$ of a DFA $\langle Q, \Sigma, \delta, q_0, F \rangle$ is said to be inaccessible if for all $x \in \Sigma^*$ it is the case the $\delta^*(s, x) \neq p$. 
Definition
The states $p$ and $p'$ of a DFA $\langle Q, \Sigma, \delta, q_0, F \rangle$ are said to be **distinguishable** if for some $x \in \Sigma^*$

$$\delta^*(p, x) \in F \text{ xor } \delta^*(q, x) \in F$$

Definition
The states $p$ and $p'$ of a DFA $\langle Q, \Sigma, \delta, q_0, F \rangle$ are said to be **indistinguishable**, written $p \approx q$, if for all $x \in \Sigma^*$

$$\delta^*(p, x) \in F \leftrightarrow \delta^*(q, x) \in F$$
Minimization Algorithm

For a DFA \( \langle Q, \Sigma, \delta, q_0, F \rangle \)

1. Remove inaccessible states
2. For every two different states, determine whether they are distinguishable
3. Collapse indistinguishable states
Marking Algorithm

For a DFA without inaccessible states $\langle Q, \Sigma, \delta, q_0, F \rangle$

1. Mark all unordered pairs $\{p, q\} \in Q \times Q$ as indistinguishable
2. Mark $\{p, q\}$ as distinguishable, if $p \in F$ xor $q \in F$
3. Repeat until no further changes: mark $\{p, q\}$ as distinguishable, if $\{\delta(p, a), \delta(q, a)\}$ is distinguishable for some $a \in \Sigma$. 
DFA Minimization
An Example
Combining Indistinguishable States
Linz 6th, Example 2.18, page 69
Minimize a DFA

Find a DFA equivalent to the one below with the minimum number of states.

An example DFA. Linz 6th, Figure 2.18, page 69.
Minimize a DFA

- DFA Minimization 8 Linz 6th, Example 2.18—3 of 22
Minimize a DFA

DFA Minimization
Minimize a DFA

\[
\begin{array}{c}
A \\
\approx B \\
\times C \\
D \\
E
\end{array}
\]
Minimize a DFA

\[ A \approx B \times C \]

\[ D \]

\[ E \]
Minimize a DFA

\[
\begin{align*}
A & \approx B \\
\times \times & C \\
\approx & D \\
& E
\end{align*}
\]
Minimize a DFA

A \approx B \times C \approx D \approx E

DFA Minimization 13
Linz 6th, Example 2.18—8 of 22
Minimize a DFA

\[
\begin{array}{c}
A \\
\approx B \\
\times \times C \\
\approx \approx \times D \\
E
\end{array}
\]
Minimize a DFA

- DFA Minimization
  - Example 2.18—10 of 22
Minimize a DFA

A

B

C

D

E

\[ A \approx B \times C \approx D \times E \]
Minimize a DFA

A
B
C
D
E

\[ A \approx B \times C \approx D \times E \]

DFA Minimization
Linz 6th, Example 2.18—12 of 22
Minimize a DFA

\[
\begin{array}{c|cc}
A & B & C \\
\approx & B & C \\
\times & \times & D \\
\times & \times & \approx & \times & E
\end{array}
\]
Minimize a DFA

\{A, B, D\}, \{C, E\}
Minimize a DFA

All pairs which were marked distinguished remain distinguish from then on out. All the others are reexamined to see if they might now be distinguishable.

\[
\begin{array}{c|cc}
A & B & C \times D \\
\hline
B & \times & C \\
C & \times & D \\
D & \times & \times & E \\
\end{array}
\]
Minimize a DFA

All pairs which were marked distinguished remain distinguish from then on out. All the others are reexamined to see if they might now be distinguishable.

\[
\begin{array}{c}
A \\
\times B \\
\times \times C \\
\times \times D \\
\times \times \times E
\end{array}
\]
Minimize a DFA

All pairs which were marked distinguished remain distinguish from then on out. All the others are reexamined to see if they might now be distinguishable.
Minimize a DFA

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Minimize a DFA

All pairs which were marked distinguished remain distinguish from then on out. All the others are reexamined to see if they might now be distinguishable.

\[
\begin{array}{cc}
A & B \\
\times & \times \\
\times & \times & \times & C \\
\times & \approx & \times & D \\
\times & \times & \times & \times & \times & \times & E
\end{array}
\]
Minimize a DFA

All pairs which were marked distinguished remain distinguish from then on out. All the others are reexamined to see if they might now be distinguishable.

\[ \{A\}, \{B, D\}, \{C\}, \{E\} \]
Minimize a DFA

Merge states $B$ and $D$.
Minimize a DFA (Solution)

\[ Q \quad \Sigma \quad Q \]
\[
\begin{array}{ccc}
A & a & BD \\
A & b & BD \\
BD & a & C \\
BD & b & E \\
C & a & BD \\
C & b & E \\
E & a & E \\
E & b & E \\
\end{array}
\]
Design a DFA over \( \{a, b\} \) containing at least three occurrences of the three consecutive \( b \)'s, overlapping permitted.