3. Graphs

- basic definitions and applications
- graph connectivity and graph traversal
- testing bipartiteness
- connectivity in directed graphs
- DAGs and topological ordering

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**Undirected graphs**

**Notation.** $G = (V, E)$

- $V =$ nodes.
- $E =$ edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|, m = |E|$. 

$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E = \{1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6, 7-8\}$

$m = 11, n = 8$
The evolution of FCC lobbying coalitions

The Spread of Obesity in a Large Social Network over 32 Years

Some graph applications

<table>
<thead>
<tr>
<th>graph</th>
<th>node</th>
<th>edge</th>
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<tbody>
<tr>
<td>communication</td>
<td>telephone, computer</td>
<td>fiber optic cable</td>
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<td>circuit</td>
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<td>wire</td>
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<tr>
<td>mechanical</td>
<td>joint</td>
<td>rod, beam, spring</td>
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<td>financial</td>
<td>stock, currency</td>
<td>transactions</td>
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<td>transportation</td>
<td>street intersection, airport</td>
<td>highway, airway route</td>
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<td>class C network</td>
<td>connection</td>
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<td>game</td>
<td>board position</td>
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<tr>
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<td>person, actor</td>
<td>friendship, movie cast</td>
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<tr>
<td>neural network</td>
<td>neuron</td>
<td>synapse</td>
</tr>
<tr>
<td>protein network</td>
<td>protein</td>
<td>protein-protein interaction</td>
</tr>
<tr>
<td>molecule</td>
<td>atom</td>
<td>bond</td>
</tr>
</tbody>
</table>

Graph representation: adjacency matrix

Adjacency matrix. An n-by-n matrix with $A_{uv} = 1$ if $(u, v)$ is an edge.

- Two representations of each edge.
- Space proportional to $n^2$.
- Checking if $(u, v)$ is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.
**Graph representation: adjacency lists**

**Adjacency lists.** Node indexed array of lists.
- Two representations of each edge.
- Space is $\Theta(m + n)$.
- Checking if $(u, v)$ is an edge takes $O(\text{degree}(u))$ time.
- Identifying all edges takes $\Theta(m + n)$ time.

**Paths and connectivity**

**Def.** A path in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair $v_i, v_{i+1}$ is joined by an edge in $E$.

**Def.** A path is **simple** if all nodes are distinct.

**Def.** An undirected graph is **connected** if for every pair of nodes $u$ and $v$, there is a path between $u$ and $v$.

**Cycles**

**Def.** A cycle is a path $v_1, v_2, \ldots, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k - 1$ nodes are all distinct.

**Trees**

**Def.** An undirected graph is a tree if it is connected and does not contain a cycle.

**Theorem.** Let $G$ be an undirected graph on $n$ nodes. Any two of the following statements imply the third.
- $G$ is connected.
- $G$ does not contain a cycle.
- $G$ has $n - 1$ edges.
Rooted trees

Given a tree $T$, choose a root node $r$ and orient each edge away from $r$.

Importance. Models hierarchical structure.

Phylogeny trees

Describe evolutionary history of species.

GUI containment hierarchy

Describe organization of GUI widgets.
**Connectivity**

**s-t connectivity problem.** Given two nodes \( s \) and \( t \), is there a path between \( s \) and \( t \)?

**s-t shortest path problem.** Given two nodes \( s \) and \( t \), what is the length of the shortest path between \( s \) and \( t \)?

**Applications.**
- Friendster.
- Maze traversal.
- Kevin Bacon number.
- Fewest number of hops in a communication network.

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**Breadth-first search**

**BFS intuition.** Explore outward from \( s \) in all possible directions, adding nodes one "layer" at a time.

**BFS algorithm.**
- \( L_0 = \{ s \} \).
- \( L_1 \) = all neighbors of \( L_0 \).
- \( L_2 \) = all nodes that do not belong to \( L_0 \) or \( L_1 \), and that have an edge to a node in \( L_1 \).
- \( L_{i+1} \) = all nodes that do not belong to an earlier layer, and that have an edge to a node in \( L_i \).

**Theorem.** For each \( i \), \( L_i \) consists of all nodes at distance exactly \( i \) from \( s \). There is a path from \( s \) to \( t \) iff \( t \) appears in some layer.

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**Breadth-first search: analysis**

**Theorem.** The above implementation of BFS runs in \( O(m + n) \) time if the graph is given by its adjacency representation.

**Pf.**
- Easy to prove \( O(n^2) \) running time:
  - at most \( n \) lists \( L[i] \)
  - each node occurs on at most one list; for loop runs \( \leq n \) times
  - when we consider node \( u \), there are \( \leq n \) incident edges \((u, v)\), and we spend \( O(1) \) processing each edge

- Actually runs in \( O(m + n) \) time:
  - when we consider node \( u \), there are \( \text{degree}(u) \) incident edges \((u, v)\)
  - total time processing edges is \( \sum_{(u,v)} \text{degree}(u) = 2m \).

  Each edge \((u, v)\) is counted exactly twice in sum: once in degree(u) and once in degree(v).
**Connected component**

**Connected component.** Find all nodes reachable from \( s \).

![Graph of connected component](image)

Connected component containing node \( 1 = \{ 1, 2, 3, 4, 5, 6, 7, 8 \} \).

**Flood fill**

**Flood fill.** Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

![Flood fill example](image)

**Theorem.** Upon termination, \( R \) is the connected component containing \( s \).

- BFS = explore in order of distance from \( s \).
- DFS = explore in a different way.
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**Bipartite graphs**

**Def.** An undirected graph $G = (V, E)$ is bipartite if the nodes can be colored blue or white such that every edge has one white and one blue end.

**Applications.**
- Stable marriage: men = blue, women = white.
- Scheduling: machines = blue, jobs = white.

**Testing bipartiteness**

Many graph problems become:
- Easier if the underlying graph is bipartite (matching).
- Tractable if the underlying graph is bipartite (independent set).

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.

**An obstruction to bipartiteness**

**Lemma.** If a graph $G$ is bipartite, it cannot contain an odd length cycle.

**Pf.** Not possible to 2-color the odd cycle, let alone $G$. 
Bipartite graphs

**Lemma.** Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

\[ \text{Case (i)} \]

\[ \text{Case (ii)} \]

\[ \text{Pf.} \]

(ii) \begin{itemize}
  \item Suppose $(x, y)$ is an edge with $x, y$ in same level $L_j$.
  \item Let $z = \text{lca}(x, y)$ = lowest common ancestor.
  \item Let $L_z$ be level containing $z$.
  \item Consider cycle that takes edge from $x$ to $y$, then path from $y$ to $z$, then path from $z$ to $x$.
  \item Its length is $1 + (j-i) + (j-i)$, which is odd. \end{itemize}

\[ \text{Corollary.} \] A graph $G$ is bipartite iff it contain no odd length cycle.
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### Directed graphs

**Notation.** $G = (V, E)$.
- Edge $(u, v)$ leaves node $u$ and enters node $v$.

**Ex.** Web graph: hyperlink points from one web page to another.
- Orientation of edges is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

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**World wide web**

**Web graph.**
- Node: web page.
- Edge: hyperlink from one page to another (orientation is crucial).
- Modern search engines exploit hyperlink structure to rank web pages by importance.

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**Road network**

**Vertex = intersection; edge = one-way street.**
The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005

Some directed graph applications

<table>
<thead>
<tr>
<th>directed graph</th>
<th>node</th>
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<tbody>
<tr>
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<td>inherits from</td>
</tr>
<tr>
<td>control flow</td>
<td>code block</td>
<td>jump</td>
</tr>
</tbody>
</table>

Directed reachability. Given a node s, find all nodes reachable from s.

Directed s-t shortest path problem. Given two node s and t, what is the length of the shortest path from s and t?

Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

Ecological food web

Food web graph.
- Node = species.
- Edge = from prey to predator.

### Strong connectivity

**Def.** Nodes \(u\) and \(v\) are **mutually reachable** if there is both a path from \(u\) to \(v\) and also a path from \(v\) to \(u\).

**Def.** A graph is **strongly connected** if every pair of nodes is mutually reachable.

**Lemma.** Let \(s\) be any node. \(G\) is strongly connected iff every node is reachable from \(s\), and \(s\) is reachable from every node.

**Pf.** \(\Rightarrow\) Follows from definition.

**Pf.** \(\Leftarrow\) Path from \(u\) to \(v\): concatenate \(u \rightarrow s\) path with \(s \rightarrow v\) path.
Path from \(v\) to \(u\): concatenate \(v \rightarrow s\) path with \(s \rightarrow u\) path. \(\blacksquare\)

\(\checkmark\) ok if paths overlap

### Strong components

**Def.** A **strong component** is a maximal subset of mutually reachable nodes.

**Theorem.** [Tarjan 1972] Can find all strong components in \(O(m + n)\) time.

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**Section 3.6**

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**Strong connectivity: algorithm**

**Theorem.** Can determine if \(G\) is strongly connected in \(O(m + n)\) time.

**Pf.**
- Pick any node \(s\).
- Run BFS from \(s\) in \(G\).
- Run BFS from \(s\) in \(G_{\text{reverse}}\).
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. \(\blacksquare\)
**Directed acyclic graphs**

**Def.** A DAG is a directed graph that contains no directed cycles.

**Def.** A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge $(v_i, v_j)$ we have $i < j$.

**Precedence constraints.** Edge $(v_i, v_j)$ means task $v_i$ must occur before $v_j$.

**Applications.**
- Course prerequisite graph: course $v_i$ must be taken before $v_j$.
- Compilation: module $v_i$ must be compiled before $v_j$.
- Pipeline of computing jobs: output of job $v_i$ needed to determine input of job $v_j$.

**Directed acyclic graphs**

**Lemma.** If $G$ has a topological order, then $G$ is a DAG.

**Pf.** [by contradiction]
- Suppose that $G$ has a topological order $v_1, v_2, \ldots, v_n$ and that $G$ also has a directed cycle $C$. Let’s see what happens.
- Let $v_i$ be the lowest-indexed node in $C$, and let $v_j$ be the node just before $v_i$; thus $(v_i, v_j)$ is an edge.
- By our choice of $i$, we have $i < j$.
- On the other hand, since $(v_j, v_i)$ is an edge and $v_1, v_2, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction.

**Directed acyclic graphs**

**Lemma.** If $G$ has a topological order, then $G$ is a DAG.

**Q.** Does every DAG have a topological ordering?

**Q.** If so, how do we compute one?
Directed acyclic graphs

**Lemma.** If $G$ is a DAG, then $G$ has a node with no entering edges.

**Pf.** [by contradiction]
- Suppose that $G$ is a DAG and every node has at least one entering edge. Let’s see what happens.
- Pick any node $v$, and begin following edges backward from $v$. Since $v$ has at least one entering edge $(u, v)$ we can walk backward to $u$.
- Then, since $u$ has at least one entering edge $(x, u)$, we can walk backward to $x$.
- Repeat until we visit a node, say $w$, twice.
- Let $C$ denote the sequence of nodes encountered between successive visits to $w$. $C$ is a cycle. □

**Directed acyclic graphs**

**Lemma.** If $G$ is a DAG, then $G$ has a topological ordering.

**Pf.** [by induction on $n$]
- Base case: true if $n = 1$.
- Given DAG on $n > 1$ nodes, find a node $v$ with no entering edges.
- $G - \{v\}$ is a DAG, since deleting $v$ cannot create cycles.
- By inductive hypothesis, $G - \{v\}$ has a topological ordering.
- Place $v$ first in topological ordering; then append nodes of $G - \{v\}$ in topological order. This is valid since $v$ has no entering edges. □

Topological sorting algorithm: running time

**Theorem.** Algorithm finds a topological order in $O(m + n)$ time.

**Pf.**
- Maintain the following information:
  - $\text{count}(w)$ = remaining number of incoming edges
  - $S$ = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete $v$
  - remove $v$ from $S$
  - decrement $\text{count}(w)$ for all edges from $v$ to $w$;
    and add $w$ to $S$ if $\text{count}(w)$ hits 0
  - this is $O(1)$ per edge □