**Linear Programming**

Optimize a linear function subject to linear inequalities.

**Linear Programming I**

- A refreshing example
- Standard form
- Fundamental questions
- Geometry
- Algebra
- Simplex algorithm

**Linear Programming**

Optimize a linear function subject to linear inequalities.

**Linear Programming I**

- A refreshing example
- Standard form
- Fundamental questions
- Geometry
- Algebra
- Simplex algorithm

**Brewery Problem**

**Objective Function**

- Maximize profit
- Subject to constraints

**Brewery Problem**

**Feasible Region**

- Convex feasible region
- Corner points
- Vertices of feasible region

**Brewery Problem**

**Geometry**

- Regardless of objective function coefficients, an optimal solution occurs at a vertex.

**Linear Programming**

- Linear programming: Optimize a linear function subject to linear inequalities.

**Brewery Problem**

- Converting to Standard Form
- Original input
- Standard form
- Equivalent forms
- Easy to convert variants to standard form
- Less than or equal to:
- Greater than or equal to:
- Unrestricted to nonnegative:

**Fundamental Questions**

LP. For a $\mathbb{C}^n$, $A \in \mathbb{C}^{m \times n}$, $b \in \mathbb{C}^m$, $c \in \mathbb{C}^n$, does there exist $x \in \mathbb{C}^n$ such that:

- Is LP in NP?
- Is LP in P? (P)

**Brewery Problem**

- Feasible region
- Corner points
- Vertices of feasible region

**Brewery Problem**

- Objective function
- Regardless of objective function coefficients, an optimal solution occurs at a vertex.

**Linear Programming**

- Linear programming: Optimize a linear function subject to linear inequalities.

**Brewery Problem**

- Feasible region
- Corner points
- Vertices of feasible region

**Brewery Problem**

- Objective function
- Regardless of objective function coefficients, an optimal solution occurs at a vertex.
Convexity

Convex set. If two points x and y are in the set, then so is 
\[ \lambda x + (1-\lambda)y \] for 0 ≤ \lambda ≤ 1.

Vertex. A point is in the set that can't be written as a strict convex combination of two distinct points in the set.

Purification

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

PE.
- Assume \( x \) is an optimal solution that is not a vertex.
- There exist direction \( d \) such that \( x + \lambda d \) hits \( x \) (for some \( \lambda > 0 \)).
- \( x + \lambda d \) is feasible since \( d \) is an extreme ray.
- \( x + \lambda d \) has one more zero component than \( x \).

Case 1. \( x + \lambda d \) is feasible for all \( \lambda > 0 \).
- Add \( \lambda d \) to \( x \) until first new component of \( x + \lambda d \) hits \( x \).
- \( x + \lambda d \) is feasible since \( d \) is an extreme ray and \( x + \lambda d \) is feasible.
- \( x + \lambda d \) has one more zero component than \( x \).
- \( x + \lambda d \) is a vertex.

Case 2. \( x + \lambda d \) is not feasible for all \( \lambda > 0 \).
- \( x + \lambda d \) is not feasible for all \( \lambda > 0 \), since \( d \) is an extreme ray.
- \( x + \lambda d \) has one more zero component than \( x \).
- \( x + \lambda d \) is a vertex.

Intuition. The vertex in \( \mathbb{R}^n \) is uniquely specified by a linearly independent equations.

Linear Programming I

Convexity

Convex set. If two points x and y are in the set, then so is 
\[ \lambda x + (1-\lambda)y \] for 0 ≤ \lambda ≤ 1.

Vertex. A point is in the set that can't be written as a strict convex combination of two distinct points in the set.

Purification

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

PE.
- Assume \( x \) is an optimal solution that is not a vertex.
- There exist direction \( d \) such that \( x + \lambda d \) hits \( x \) (for some \( \lambda > 0 \)).
- \( x + \lambda d \) is feasible since \( d \) is an extreme ray.
- \( x + \lambda d \) has one more zero component than \( x \).

Case 1. \( x + \lambda d \) is feasible for all \( \lambda > 0 \).
- Add \( \lambda d \) to \( x \) until first new component of \( x + \lambda d \) hits \( x \).
- \( x + \lambda d \) is feasible since \( d \) is an extreme ray and \( x + \lambda d \) is feasible.
- \( x + \lambda d \) has one more zero component than \( x \).
- \( x + \lambda d \) is a vertex.

Case 2. \( x + \lambda d \) is not feasible for all \( \lambda > 0 \).
- \( x + \lambda d \) is not feasible for all \( \lambda > 0 \), since \( d \) is an extreme ray.
- \( x + \lambda d \) has one more zero component than \( x \).
- \( x + \lambda d \) is a vertex.

Intuition. The vertex in \( \mathbb{R}^n \) is uniquely specified by a linearly independent equations.
Simplex Algorithm: Pivot 1

Q. Why pivot on column 2 (or 3)?
   A. Each unit increase in \( A \) increases objective value by \$23.50.

Q. Why pivot on row 2?
   A. Ensures feasibility by ensuring \( B = 0 \).

Simplex Algorithm: Optimality

Q. When to stop pivoting?
   A. When all coefficients in top row are nonpositive.

Q. Why minimizing solution optimal?
   A. Any feasible solution satisfies system of equations in tableau.

Simplex Tableaux: Matrix Form

Initial simplex tableau,

\[
\begin{align*}
\text{max } Z & \quad \text{s.t. } \quad \begin{cases}
3x + 2y + z & = 900 \\
x + 2y + z & = 600 \\
x + y + z & = 400 \\
\end{cases}
\end{align*}
\]

Thus, optimal objective value \( Z \geq 600 \).

Current BFS has value \( 600 \) — optimal.

Simplex tableau corresponding to basis \( B \).

\[
\begin{align*}
\text{z} & = 3x + 2y + z = 900 \\
x & = 0 \\
y & = 0 \\
z & = 0
\end{align*}
\]

Simplex Algorithm: Pivot 2

Substitute: \( A = 3x + 2y + z \) and \( A = 900 \).

Q. Why pivot on column 3?
   A. Each unit increase in \( y \) increases objective value by \$50.50.

Q. Why pivot on row 1?
   A. Ensures feasibility by ensuring \( C = 0 \).

Simplex Algorithm: Corner Cases

Q. What if min ratio test fails?
   A. No basis can be found.

Q. How to find initial basis?
   A. BPS.

Q. How to guarantee termination?
   A. No cycling.

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.

Simplex Algorithm: Degeneracy

Degenerate pivot. Min ratio = 0.

Lexicographic Rule

Intuition. No degeneracy = no cycling.

Perturbed problem.

\[
\begin{align*}
\text{P'}: \quad & \text{max } Z' \\
\text{subject to } & \quad \begin{cases}
x + y & = 1 \quad \text{where } x + y = 1 \\
y & = 0 \\
\end{cases}
\end{align*}
\]

Claim. In perturbed problem, \( x = \frac{1}{2} + \epsilon \) is always nonzero.

Remark. \( \frac{1}{2} \) component of \( x \) is a (nonzero) linear combination of the components of \( x = y \) contains at least one of the \( x \) terms.

Lexicographic rule. Apply perturbation virtually by manipulating \( \epsilon \) symbolically.

\[
\begin{align*}
15 x + 15 y & = 1500 + \epsilon \\
15 x + 15 y & = 1500 + \epsilon
\end{align*}
\]

Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints.

Unboundedness

Q. What happens if min ratio test fails?
   A. No basis can be found.

Q. How to find initial basis?
   A. BPS.

Q. How to guarantee termination?
   A. No cycling.

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.

Degenerate pivot. Min ratio = 0.

Lexicographic Rule

Intuition. No degeneracy = no cycling.

Perturbed problem.

\[
\begin{align*}
\text{P'}: \quad & \text{max } Z' \\
\text{subject to } & \quad \begin{cases}
x + y & = 1 \quad \text{where } x + y = 1 \\
y & = 0 \\
\end{cases}
\end{align*}
\]

Claim. In perturbed problem, \( x = \frac{1}{2} + \epsilon \) is always nonzero.

Remark. \( \frac{1}{2} \) component of \( x \) is a (nonzero) linear combination of the components of \( x = y \) contains at least one of the \( x \) terms.

Lexicographic rule. Apply perturbation virtually by manipulating \( \epsilon \) symbolically.

\[
\begin{align*}
15 x + 15 y & = 1500 + \epsilon \\
15 x + 15 y & = 1500 + \epsilon
\end{align*}
\]

Commercial solvers can solve LPs with millions of variables and tens of thousands of constraints.

Unboundedness

Q. What happens if min ratio test fails?
   A. No basis can be found.

Q. How to find initial basis?
   A. BPS.

Q. How to guarantee termination?
   A. No cycling.

Simplex Algorithm: Degeneracy

Degeneracy. New basis, same vertex.

Degenerate pivot. Min ratio = 0.