CSE 1400 Applied Discrete Mathematics
Logarithms and Exponents

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Logarithms 1

Properties of Logarithms 2

Change of Base Formula 3

Computing the Logarithm of a Number 4

Calculating the Numerals in a Number 4

Problems on Logarithms and Exponentials 5

Abstract

Logarithmic functions and their inverses, exponential functions occur in many application areas: annuities and loans, growth and decay, seismic modeling, and algorithm analysis.

Logarithms

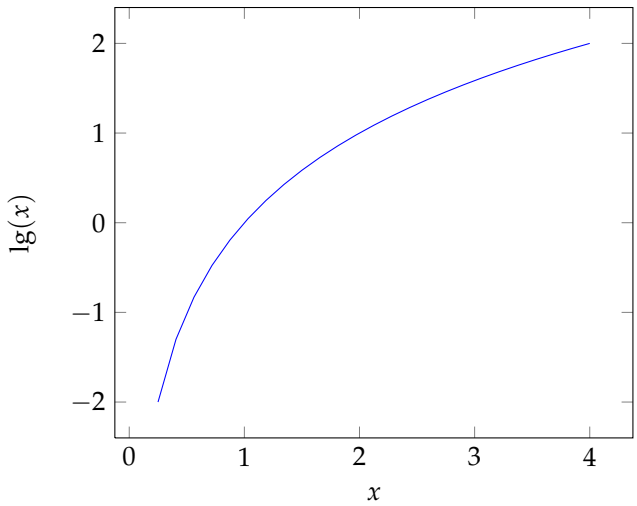
The logarithm base e is called the natural logarithm. Using calculus it can be established that

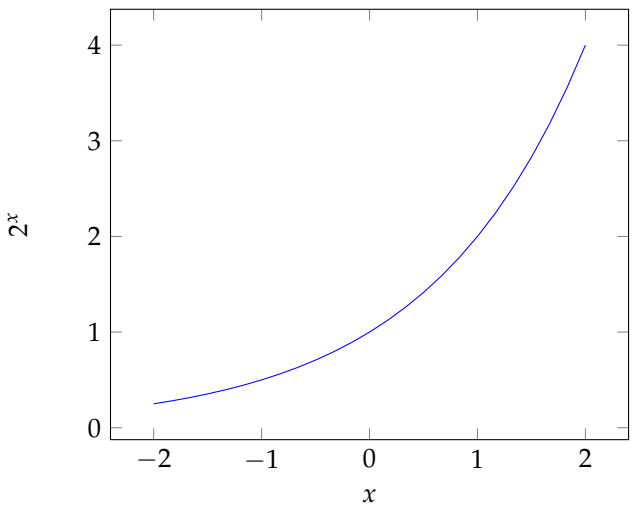
$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}$$

where the approximation grows more accurate as the degree n of the approximating MacLaurin polynomial increases. The inverse of the natural logarithm is the exponential base e

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

MacLaurin's approximation, together with Horner's rule provides an algorithm for approximating the logarithmic and exponential functions at a value $x = a$.





Properties of Logarithms

Logarithms have several useful properties. They were invented to simplify computations. They convert multiplication into addition, division into subtraction, and exponentiation into multiplication. These are the easily established rules.

1. Log of a product is the sum of logs

$$\ln ab = \ln a + \ln b$$

because

$$e^a e^b = e^{a+b}$$

2. Log of a quotient is the difference of logs

$$\ln \frac{a}{b} = \ln a - \ln b$$

because

$$\frac{e^a}{e^b} = e^{a-b}$$

3. Log of a power is the power times the log

$$\ln a^b = b \ln a$$

because

$$(e^a)^b = e^{ab}$$

In computer science, decisions can be reduced to "yes" or "no" answers. Because of this the binary logarithm

$$\lg x = \log_2 x$$

plays an important role.

x	1	2	4	8	16	32	64	128	256	512	1024
$\lg x$	0	1	2	3	4	5	6	7	8	9	10

Change of Base Formula

There are three primary logarithm bases: 10, e, and 2.

- The log of x base 10 is written $\log x$ and called the common logarithm.
- The log of x base e is written $\ln x$ and called the natural logarithm.
- The log of x base 2 is written $\lg x$ and called the binary logarithm.

Luckily, logarithms in one base can be easily related to logarithms in another base. Consider how the log base β of x can be evaluated using the log base ξ. Read the equations from left-to-right and top-to-bottom.

$$y = \log_\beta x \quad \beta^y = x \qquad \log_\alpha \beta^y = \log_\alpha x$$
$$y \log_\alpha \beta = \log_\alpha x \quad y = \frac{\log_\alpha x}{\log_\alpha \beta} \quad \log_\beta x = \frac{\log_\alpha x}{\log_\alpha \beta}$$

That is, the log of x base β can be written as the log of x base α divided by the log of β base α:

$$\log_\beta x = \frac{\log_\alpha x}{\log_\alpha \beta}$$

For instance

$$\log x = \frac{\lg x}{\lg 10} = \frac{\ln x}{\ln 10}$$
$$\ln x = \frac{\lg x}{\lg e} = \frac{\log x}{\log e}$$
$$\lg x = \frac{\log x}{\log 2} = \frac{\ln x}{\ln 2}$$

Computing the Logarithm of a Number

Using the properties of logarithms and the change of base formula, the log of some numbers can be easily calculated. Here are some examples.

$$\lg 128 = \lg 2^7 = 7$$
$$\lg \sqrt[3]{128} = \lg 2^{7/3} = 7/3$$
$$\log_8 \sqrt[7]{64} = \log_8 2^{6/7} = \frac{6 \lg 2}{7 \lg 8} = \frac{2}{7}$$

It is sometimes easy to compute a good approximation to the value of a logarithm. For instance, since $3^2 \approx 2^3$ you can compute the approximation

$$\log_3 8 = 3 \log_3 2 \approx \log_3 9 = 2$$

or

$$\log_3 2 \approx= 2/3$$

Calculating the Numerals in a Number

It takes m digits to write a natural number n such that

$$10^{m-1} \leq n < 10^m$$

For instance,

- Numbers n from 1 to 9 ($10^0 \leq n < 10^1$) can be written in using 1 digit.
- Numbers n from 10 to 99 ($10^1 \leq n < 10^2$) can be written in using 2 digit.
- Numbers n from 100 to 999 ($10^2 \leq n < 10^3$) can be written in using 3 digit.

Therefore, if $10^{m-1} \leq n < 10^m$, then taking common logarithms,

$$m - 1 \leq \log n < m$$

which shows that m is equal to the floor of $\log n$ plus 1.

The floor function is studied in another section of this course.

$$m = \lfloor \log n \rfloor + 1$$

It takes $m = \lfloor \log n \rfloor + 1$ digits to write a natural number n.

By an analogous argument, it takes m bits to write a natural number n such that

$$2^{m-1} \leq n < 2^m$$

For instance,

- Numbers n from 1 to 1 ($2^0 \leq n < 2^1$) can be written in using 1 bit.
- Numbers n from 2 to 3 ($2^1 \leq n < 2^2$) can be written in using 2 bit.
- Numbers n from 4 to 7 ($2^2 \leq n < 2^3$) can be written in using 3 bit.

Therefore, if $2^{m-1} \leq n < 2^m$, then taking logarithms base 2,

$$m - 1 \leq \lg n < m$$

which shows that m is equal to the floor of $\lg n$ plus 1.

$$m = \lfloor \lg n \rfloor + 1$$

It takes $m = \lfloor \lg n \rfloor + 1$ bits to write a natural number n.

Problems on Logarithms and Exponentials

1. Answer `True` or `False`. Explain your answer.

   (a) $2^a 2^b = 2^{a+b}$.
   (b) $2^a 2^b = 2^{ab}$.
   (c) $2^n 2^n = 2^{2n}$.
   (d) $2^n 2^n = 2^{n^2}$.
   (e) $(2^a)^b = 2^{ab}$.
   (f) $(2^a)^b = 2^{a^b}$.
   (g) $(2^n)^n = 2^{n^2}$.
   (h) $(2^n)^n = 2^{n^n}$.

2. Evaluate the logarithm base 2 function $\lg x$ at the given value for x.

   (a) $\lg(256)$
   (b) $\lg(0.25)$
   (c) $\lg(0.125)$
   (d) $\lg(1/1024)$
   (e) $\lg(\sqrt[3]{16})$
   (f) $\lg(\sqrt[5]{8})$
   (g) $\lg \sqrt[3]{32}$
   (h) $\log_{16} \sqrt[3]{32}$
   (i) $\lg(2^{n!})$

3. In binary notation, how many bits are required to name a natural number n when $2^3 \leq n < 2^4$?
4. In binary notation, how many bits are required to name a natural number n
5. `True` or `False`: The logarithm (base 2) function $\lg(x)$ maps the open interval $(0, \infty)$ onto the set of real numbers. Explain your answer.
6. `True` or `False`: The logarithm (base 2) function $\lg(x)$ is NOT one-to-one. Explain your answer.
7. `True` or `False`: Since $2^{10} = 1024$ is approximately equal to $10^3 = 1000$, the log base 2 of 10 is approximately equal to 3 and $1/3$. Explain your answer.
8. Use the fact that $2^7 = 128$ is approximately equal to $5^3 = 125$ to approximate the value of $\log_5(2)$.
9. Given that $\ln 2 \approx 0.693147$ and $\log 2 \approx 0.301030$ Show that

$$\lg x \approx \ln x + \log x$$

Specifically, that the error is less than 1% in that

$$\left| \frac{\ln x + \log x}{\lg x} - 1 \right| < 0.01$$

10. For each problem below, find a constant $c \in \mathbb{R}$ such that the equation is an identity.

(a) $\lg(x) = c \ln(x)$
(b) $\ln(x) = c \lg(x)$
(c) $\lg(x) = c \log(x)$
(d) $\log(x) = c \lg(x)$
(e) $\lg(x) = c \log_b(x)$
(f) $\log_b(x) = c \lg(x)$

11. Consider the proposition: $\log_a(b) \cdot \log_b(a) = 1$. Is the position `True` or `False`. You may assume a and b are natural numbers greater than or equal to 2. Explain your answer.
12. Use the "sum of logs is log of a product rule" to write the sum

$$\sum_{1 \leq k \leq n} \lg k$$

as a logarithm that involves a factorial.

13. A solution's pH measures its acidity or alkalinity, with 7 called neutral. In particular,

$$\mathrm{pH} = -\log_{10}(\mathrm{H_3O^+})$$

where $\mathrm{H_3O^+}$ is the concentration of hydronium ions in the solution. The pH scale is logarithmic: When pH increases by 1, alkalinity increases 10-fold. Likewise, when pH decreases by 1, acidity increases 10-fold. Pretend the average pH of sea water will decrease by 0.5 in 100 years. By what factor will the acidity sea water increase?

14. If a principle of P dollars is invested at an interest rate r compounded n times per year, then the amount after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Pretend $P = \$1$ is invested at a 10% rate compounded quarterly. How many years t will go by before $A = \$10$?