CSE 1400 Applied Discrete Mathematics
Permutations

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Abstract

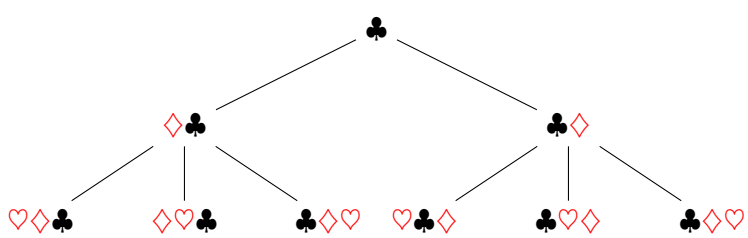
A permutation is a one-to-one function from a set onto itself.

Permutations

A PERMUTATION IS A FUNCTION THAT REARRANGES THE ORDER OF TERMS IN A SEQUENCE. It is useful to study a few small examples. Consider the suits in a deck of playing cards: clubs ♣, diamonds ♢, hearts ♡, and spades ♠.

In computing practice, sorting a group of objects into a preferred order is a fundamental operation. Sorting algorithms perform a sequence of permutations on the objects, each bringing them closer to the preferred order.

There are $2! = 2$ permutations of two THINGS .



There are $3! = 6$ permutations of three THINGS .

Starting with a ♣, after picking up a ♢, place it before or after the ♣. If you next draw a ♡ it can be place before, in the middle, or after the already permuted pairs.

Imagine inserting a ♠ into one of the already arranged suits, say ♡♣♢. There are four places where the ♠ can be inserted: first, second, third, or fourth. Reasoning like this it is not difficult to observe there are

♠♡♣♢, ♡♠♣♢, ♡♣♠♢, ♡♣♢♠

$$4! = 4 \cdot 6 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

permutations on 4 THINGS .

Let $\mathbb{A}$ be a ith n elements. The symbol for the count of ways to permute the elements of $\mathbb{A}$ is $n!$ and pronounced "n factorial." This COUNT OF PERMUTATIONS can be computed by evaluating the product

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

called n factorial.

Cyclic Notation

UNDER A PERMUTATION A THING IN SPOT n "GOES TO" SPOT m. Cyclic notation describes "goes to."

Consider the permutation "shift by 2"

$$[0, 2, 4, \cdots, 24][1, 3, 5, \cdots, 25]$$

on the English alphabet applied to the characters in the statement

a man a plan a canal panama

c ocp c rncp c ecpcn rcpcoc

In computing practice, you may need to save the THING is spot m before overwriting it with the THING from spot n.

For instance, under the permutation

Original Order	0	1	2	3	4
Permuted Order	4	3	2	1	0

value 0 goes to position 4 and 4 goes to 0, 1 goes to position 3 and 3 goes to 1, and 2 stays put. In cyclic notation, the above permutation is written

$$[0, 4][1, 3][2]$$

Okay! Blank did not shift.

Permutations Re-Order a Sequence

FOR SMALL SETS EACH PERMUTATION CAN BE LISTED. Let $\mathbb{A}$ be a set with cardinality $|\mathbb{A}| = n$. There are n factorial different permutations of the elements in $\mathbb{A}$. Figure 1 shows the $3! = 6$ permutations of the elements in $\{0, 1, 2\}$ written in cyclic notation.

It is convenient to say there is a permutation of the $\emptyset$, so there is 1. That is, $0! = 1$.

The permutations on $\{0, 1, 2, 3\}$ can be defined recursively, that is, from the permutations on $\{0, 1, 2\}$. For instance, to build all 2-cycle permutations of $\{0, 1, 2, 3\}$. use the one and two-cycle permutations of $\{0, 1, 2\}$.

1. Append the cycle $[3]$ to each 1-cycle permutation of $\{0, 1, 2\}$
2. Insert new element 3 in three positions in each 2-cycle permutations of $\{0, 1, 2\}$

Using $\left[{n \atop m}\right]$ to name the count of 2-cycle permutations of a 4-element set, write

$$\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} + 2\begin{bmatrix}3\\2\end{bmatrix} = 2 + 3\cdot 3 = 11$$

These eleven permutations are shown in figure 2.

Permutations		
[0, 1, 2]	[0][1, 2]	[0][1][2]
[0, 2, 1]	[1][0, 2]	
	[2][0, 1]	

Figure 1: Cyclic notation for the 3! permutations of $\{0, 1, 2\}$.

Permutations			
[0, 1, 2, 3]	[0, 1, 2][3]	[0][1, 2][3]	[0][1][2][3]
[0, 1, 3, 2]	[0, 2, 1][3]	[1][0, 2][3]	
[0, 3, 1, 2]	[0][1, 2, 3]	[2][0, 1][3]	
[0, 2, 1, 3]	[0][1, 3, 2]	[0][1][2, 3]	
[0, 2, 3, 1]	[0, 3][1, 2]	[0][1, 3][2]	
[0, 3, 2, 1]	[1][0, 2, 3]	[0, 3][1][2]	
	[1][0, 3, 2]		
	[1, 3][0, 2]		
	[2][0, 1, 3]		
	[2][0, 3, 1]		
	[2, 3][0, 1]		

Figure 2: Cyclic notation for the 4! permutations of $\{0, 1, 2, 3\}$.

Stirling Numbers of the First Kind

THE ELEMENTS OF SET WITH CARDINALITY n CAN BE PERMUTED INTO m CYCLES IN $\left[{n \atop m}\right]$ WAYS. Stirling numbers of the first kind are defined by the recurrence equation

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ m \end{bmatrix}$$

with boundary conditions

$$\begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 0, \text{ for } n > 0$$

Check that the following arithmetic can be verified by the numbers in table 1.

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 + 3 \cdot 1$$
$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 11 + 4 \cdot 6$$
$$\begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} + 6\begin{bmatrix} 6 \\ 5 \end{bmatrix} = 85 + 6 \cdot 15$$

The notation $\left[{n \atop m}\right]$ is called *n cycle m*.

Table 1: Stirling numbers of the first kind $\left[{n \atop m}\right]$ count the permutations with m cycles of n THINGS .

Stirling Numbers of the First Kind $\left[{n \atop m}\right]$

		Cycle m								
		0	1	2	3	4	5	6	7	8
	0	1	0	0	0	0	0	0	0	0
	1	0	1	0	0	0	0	0	0	0
	2	0	1	1	0	0	0	0	0	0
	3	0	2	3	1	0	0	0	0	0
n	4	0	6	11	6	1	0	0	0	0
	5	0	24	50	35	10	1	0	0	0
	6	0	120	274	225	85	15	1	0	0
	7	0	720	1764	1624	735	175	21	1	0
	8	0	5040	13068	13132	6769	1960	322	28	1

Problems on Permutations

1. True or false: A permutation of $\mathbb{X}$ is a one-to-one function from $\mathbb{X}$ onto $\mathbb{X}$.
2. True or false: $[c, a, b]$ is a permutation of $\{a, b, c\}$.
3. True or false: $[a, a, b]$ is a permutation of $\{a, b, c\}$.
4. True or false: There are 2^n permutations of an n element set.
5. True or false: There are $n!$ permutations of an n element set.
6. Let $\mathbb{H} = \{0, 1, 2, \ldots, E, F\}$.
   (a) How many permutations can be defined on $\mathbb{H}$?
   (b) In how many ways can you choose 5 elements from $\mathbb{H}$

(c) In how many ways can you choose and permute 5 elements from $\mathbb{H}$

(d) How many permutations on $\mathbb{H}$ have 16 cycles?

(e) How many permutations on $\mathbb{H}$ have 1 cycle?

7. Let $\mathbb{X}$ be an 8-element set. How many permutations on $\mathbb{X}$ have 4 cycles? That is, what is the value of the Stirling number $\left[{8 \atop 4}\right]$ "8 cycle 4"? You may want to know row 7 of Stirling's triangle of the first kind is

	0	1	2	3	4	$\cdots$
7	0	720	1764	1624	735	$\cdots$

8. Use cyclic notation to describe the permutation $(0, 1, 2, 3)$ of the elements in the set $\{0, 1, 2, 3\}$

9. Use cyclic notation to describe the permutation $(0, 2, 4, 6, 1, 3, 5, 7)$ of the octal alphabet $\{0, 1, 2, 3, 4, 5, 6, 7\}$

10. Use cyclic notation to describe the permutation $(1, 2, 0, 7, 3, 4, 5, 6)$ of the octal alphabet $\{0, 1, 2, 3, 4, 5, 6, 7\}$