Permutations

Abstract

A permutation is a one-to-one function from a set onto itself.

Permutations

A permutation is a function that rearranges the order of terms in a sequence. It is useful to study a few small examples. Consider the suits in a deck of playing cards: clubs ♠, diamonds ♦, hearts ♥, and spades ♣.

In computing practice, sorting a group of objects into a preferred order is a fundamental operation. Sorting algorithms perform a sequence of permutations on the objects, each bringing them closer to the preferred order. There are 2! = 2 permutations of two things.

Starting with a ♠, after picking up a ♦, place it before or after the ♠. If you next draw a ♥ it can be placed before, in the middle, or after the already permuted pairs.

Imagine inserting a ♣ into one of the already arranged suits, say ♥♠♣. There are four places where the ♣ can be inserted: first, second, third, or fourth. Reasoning like this it is not difficult to observe there are

\[ 4! = 4 \cdot 6 = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \]
permutations on 4 things.

Let $A$ be a set of $n$ elements. The symbol for the count of ways to permute the elements of $A$ is $n!$ and pronounced "$n$ factorial." This count of permutations can be computed by evaluating the product

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

called $n$ factorial.

Cyclic Notation

Under a permutation a thing in spot $n$ "goes to" spot $m$.

Cyclic notation describes "goes to."

Consider the permutation "shift by 2"

$$[0, 2, 4, \ldots, 24][1, 3, 5, \ldots, 25]$$

on the English alphabet applied to the characters in the statement

da man a plan a canal panama

c ocp c rncp c ecpcn rcpcoc

Permutations Re-Order a Sequence

For small sets each permutation can be listed. Let $A$ be a set with cardinality $|A| = n$. There are $n$ factorial different permutations of the elements in $A$. Figure 1 shows the $3! = 6$ permutations of the elements in $\{0, 1, 2\}$ written in cyclic notation.

The permutations on $\{0, 1, 2, 3\}$ can be defined recursively, that is, from the permutations on $\{0, 1, 2\}$. For instance, to build all 2-cycle permutations of $\{0, 1, 2, 3\}$, use the one and two-cycle permutations of $\{0, 1, 2\}$.

1. Append the cycle $[3]$ to each 1-cycle permutation of $\{0, 1, 2\}$
2. Insert new element 3 in three positions in each 2-cycle permutations of $\{0, 1, 2\}$

Using $\binom{n}{m}$ to name the count of 2-cycle permutations of a 4-element set, write

$$\binom{4}{2} = \binom{3}{1} + 2\binom{3}{2} = 2 + 3 \cdot 3 = 11$$

These eleven permutations are shown in figure 2.
Permutations

Figure 1: Cyclic notation for the $3!$ permutations of $\{0, 1, 2\}$.

Permutations

Figure 2: Cyclic notation for the $4!$ permutations of $\{0, 1, 2, 3\}$.


Stirling Numbers of the First Kind

The elements of set with cardinality \( n \) can be permuted into \( m \) cycles in \( \binom{n}{m} \) ways. Stirling numbers of the first kind are defined by the recurrence equation

\[
\binom{n}{m} = \binom{n-1}{m-1} + (n-1) \binom{n-1}{m}
\]

with boundary conditions

\[
\binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{0} = 0, \text{ for } n > 0
\]

Check that the following arithmetic can be verified by the numbers in table 1.

\[
\begin{align*}
\binom{4}{3} &= \binom{3}{2} + \binom{3}{3} = 3 + 3 \cdot 1 \\
\binom{5}{3} &= \binom{4}{2} + \binom{4}{3} = 11 + 4 \cdot 6 \\
\binom{7}{5} &= \binom{6}{4} + \binom{6}{5} = 85 + 6 \cdot 15
\end{align*}
\]

The notation \( \binom{n}{m} \) is called \( n \) cycle \( m \).

Table 1: Stirling numbers of the first kind \( \binom{n}{m} \) count the permutations with \( m \) cycles of \( n \) things.

<table>
<thead>
<tr>
<th>Cycle ( m )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( n )</td>
<td>4</td>
<td>0</td>
<td>6</td>
<td>11</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>24</td>
<td>50</td>
<td>35</td>
<td>10</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>120</td>
<td>274</td>
<td>225</td>
<td>85</td>
<td>15</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>720</td>
<td>1764</td>
<td>1624</td>
<td>735</td>
<td>175</td>
<td>21</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>5040</td>
<td>13068</td>
<td>13132</td>
<td>6769</td>
<td>1960</td>
<td>322</td>
<td>28</td>
<td>1</td>
</tr>
</tbody>
</table>

Problems on Permutations

1. True or false: A permutation of \( X \) is a one-to-one function from \( X \) onto \( X \).
2. True or false: \([c, a, b]\) is a permutation of \([a, b, c]\).
3. True or false: \([a, a, b]\) is a permutation of \([a, b, c]\).
4. True or false: There are \( 2^n \) permutations of an \( n \) element set.
5. True or false: There are \( n! \) permutations of an \( n \) element set.
6. Let \( H = \{0, 1, 2, \ldots, E, F\} \).
   (a) How many permutations can be defined on \( H \)?
   (b) In how many ways can you choose 5 elements from \( H \)
(c) In how many ways can you choose and permute 5 elements from \( H \)?

(d) How many permutations on \( H \) have 16 cycles?

(e) How many permutations on \( H \) have 1 cycle?

7. Let \( X \) be an 8-element set. How many permutations on \( X \) have 4 cycles? That is, what is the value of the Stirling number \( \left\{ \begin{array}{c} 8 \\ 4 \end{array} \right\} \) “8 cycle 4”? You may want to know row 7 of Stirling’s triangle of the first kind is

\[
\begin{array}{ccccccc}
7 & 0 & 720 & 1764 & 1624 & 735 & \cdots \\
\end{array}
\]

8. Use cyclic notation to describe the permutation \((0, 1, 2, 3)\) of the elements in the set \( \{0, 1, 2, 3\} \).

9. Use cyclic notation to describe the permutation \((0, 2, 4, 6, 1, 3, 5, 7)\) of the octal alphabet \( \{0, 1, 2, 3, 4, 5, 6, 7\} \).

10. Use cyclic notation to describe the permutation \((1, 2, 0, 7, 3, 4, 5, 6)\) of the octal alphabet \( \{0, 1, 2, 3, 4, 5, 6, 7\} \).